A COUPLING STRATEGY FOR FREE SURFACE FLOWS

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Abstract. The aim of this paper is to present an efficient and locally accurate numerical method for the simulation of the motion of water in a complex system of channels under the hydrostatic assumption. We consider an a-priori coupling between one and two-dimensional hydrostatic free surface flow models. The reduction of the computational cost is then obtained by solving the most expensive two-dimensional model only in some parts of the computational domain. At the beginning we introduce 1D and 2D Shallow Water models with suitable energy estimations for both of them. Moreover, we discuss a domain decomposition algorithm to solve the 1D-2D coupled problem and we present some preliminary numerical results about the proposed algorithm.
1 INTRODUCTION

Free surface flows are encountered in many natural phenomena, from tidal currents, to wide water basins and many engineering and practical applications involve the study of them. For the simulation of a wide water body, a direct solution of the three dimensional Navier-Stokes equations is computationally rather expensive. However, from the physical viewpoint these type of phenomena involve specific space and time scales and, for this reason also, several simplified mathematical models was proposed in literature (see, for instance, [4], [14] and [15]). These models, derived from the Navier-Stokes equations for an incompressible free-surface flows, can be classified with respect to their space dimension (one, two and three-dimensional models) or with respect to their capability to describe the propagation of short or long waves or dispersion phenomena, or for ad hydrostatic approximation of the pressure (such as in the shallow water approximation).

Ideally, in a complex hydrographic basin, such as a network of natural and artificial channels and lagoons, one may use the more realistic (and, from the computational viewpoint, expensive) model only in the regions of the computational domain in which it is strictly necessary. The main problem is how to find the regions of the domain where the simpler and the more complex models have to be solved.

This goal can be achieved a priori or a posteriori. In the first case the choice among different models is driven by physical considerations; for instance, in an artificial channel in which the slope is assumed to be sufficiently mild and uniform a one dimensional hydrostatic model can be sufficient, while near a river bifurcation a two-dimensional model is better. Following this idea the computational domain is divided a priori in sub-regions related to different physical models in a fixed hierarchy of available models. Therefore the main problem is now to setup up the appropriate interface boundary conditions between different models. Alternatively, in the a posteriori strategy one start with the simplest model in the computational domain and, thanks to an a posteriori model error estimator, change, during the numerical simulation, the regions where each model in a fixed hierarchy is solved.

In this paper we consider the first approach. For a description of the a posteriori model adaptivity for free surface flows the reader is referred to [8]. More in detail, we consider free surface flows that are characterized by a vertical scale much smaller than the horizontal one and hence are commonly called shallow water flows. In this case, when the vertical accelerations are small (long waves phenomena), one can consider models based on the hydrostatic approximation of the pressure In particular, we deal with the coupling between inviscid 1D and 2D hydrostatic models.

2 HYDROSTATIC MODELS

In hydrodynamics when the vertical scales are much smaller than the horizontal ones the shallow water theory can be applied and the pressure can be assume hydrostatic, i.e. the pressure of the fluid is assumed to depend on the total water depth only. In
the literature, due to the great computational effort required to solve the complete three-dimensional problem and to its complexity, various one and two-dimensional numerical models, based on this assumption, have been devised. The derivation of such models is sometimes unclear and usually limited to the inviscid case (see, [15], [13] or [4]).

More recently, in [7] and in [6] a new class of viscous shallow water systems, respectively, for 1D and 2D problems, was proposed in order to taking into account rigorously the effect due the viscosity of the fluid. From the practical point of view the only difference between these models and the classical Saint Venant equations for a viscous flow is a new form of the friction term in the momentum equation that now depends on the viscosity itself. On the other hand there is a numerical evidence that the solution produced by this system exhibits a better accuracy if compared to the solution of the classical Saint-Venant equations.

2.1 The one-dimensional model

Following the previous references we consider for a one-dimensional channel \((a, b)\) with rectangular cross section the following differential system

\[
\begin{align*}
\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} &= 0 \quad \text{with } x \in (a, b), \ t > 0, \\
\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x} \left( \frac{Q^2}{A} \right) + gA \frac{\partial h}{\partial x} - \mu \frac{\partial^2 Q}{\partial x^2} + \frac{\kappa u}{1 + \frac{\kappa h}{3\mu}} &= 0 \quad \text{with } x \in (a, b), \ t > 0
\end{align*}
\]

(1)

where \(A\) is the wet area, \(Q\) the discharge, \(h\) the total water depth and \(u = Q/A\) the velocity, \(g\) is the gravity acceleration, \(\mu\) is the viscosity and \(\kappa\) a given friction coefficient. Due to the fact that the section of the channel is a rectangle we have \(h = (A - A_0)/L + h_0\), \(L\) being the width of the section, \(h_0\) denotes the constant undisturbed water depth and \(A_0\) is the corresponding section.\(^1\) We assume for the sake of simplicity that the bottom is flat. System (1) must be completed with suitable initial and boundary conditions.

It is simple to verify that, when the viscosity effects can be neglected, system (1) is hyperbolic with two distinct eigenvalues \(\lambda_1\) and \(\lambda_2\) given by

\[
\lambda_{1,2} = \frac{Q}{A} \pm c, \quad \text{where} \quad c = \sqrt{gh}.
\]

The associated eigenfunctions are provided by the characteristic variables

\[
W_{1,2} = u \pm \int_{A_0}^A \sqrt{\frac{g\tau}{L}} \frac{1}{\tau} \, d\tau = u \pm 2 \sqrt{\frac{g}{L}} \left[ \sqrt{A} - \sqrt{A_0} \right].
\]

It is possible to derive an energy estimate for the solution of system (1), neglecting the friction effects. We assume that, for any time \(t > 0\), the eigenvalues \(\lambda_1\) and \(\lambda_2\) be of

\(^1\)For general sections it is usual to assume that an algebraic relation holds between the water depth and the area as \(h = \psi(A) + h_0\), with \(\partial \psi / \partial A > 0\) and \(\psi(A_0) = 0\).
opposite sign, i.e. \( \lambda_1 > 0 \) and \( \lambda_2 < 0 \) (sub-critical flow regime and unidirectional flux), and \( A \) be strictly positive. This is the most interesting situation in view of the coupling with a two-dimensional hyperbolic model. Moreover we endow system (1) with suitable initial and boundary conditions:

\[
A(x, 0) = A^0, \quad Q(x, 0) = Q^0 \quad \text{with} \quad a \leq x \leq b, \quad (2)
\]

\[
W_1 = g_1 \quad \text{at} \quad x = a, \quad W_2 = g_2 \quad \text{at} \quad x = b.
\]

Thus, the following conservation property can be stated: for any \( T > 0 \), we have

\[
E(T) + \int_0^T Q\left((h - h_0) + \frac{1}{2g} u^2\right)_{a}^{b} dt = E(0), \quad (3)
\]

where \( E(0) \) depends only on the initial values \( A^0 \) and \( Q^0 \) and

\[
E(t) = \frac{1}{2g} \int_a^b A(x, t) u^2(x, t) dx + \frac{1}{2}(A - A_0)^2. \quad (4)
\]

Therefore, from (3) and considering homogeneous boundary conditions (i.e. \( g_1 = g_2 = 0 \)), we can conclude that the 1D model problem (1) is stable since

\[
E(T) \leq E(0),
\]

provided that \( 2\sqrt{A_0}/3 < \sqrt{A} < 2 \sqrt{A} \).

2.2 The two-dimensional model

We consider the following two-dimensional shallow water system (proposed in [6])

\[
\begin{aligned}
\frac{\partial h}{\partial t} + \nabla \cdot Q &= 0 \quad \text{for} \quad x \in \Omega, \ t > 0, \\
\frac{\partial Q}{\partial t} + \nabla \left( \frac{QQ}{h} \right) + gh \nabla h - \mu \Delta Q + \frac{\kappa Q}{h(1 + \frac{h}{3\mu})} &= 0 \quad \text{for} \quad x \in \Omega, \ t > 0,
\end{aligned} \quad (5)
\]

where \( x = (x, y)^T \), \( Q = hU \) is the unit discharge, \( U = (u, v)^T \) is the velocity field, \( h \) denotes the total water depth and \( \Omega \) is a bounded open set of \( \mathbb{R}^2 \). As in the previous Section we assume the the bottom is flat. Moreover system (5) must be completed with initial conditions for \( U \) and \( h \) and suitable boundary conditions (see, for instance, [1]).

For an unviscid flow equations (5) represent an hyperbolic system whose eigenvalues ad eigenfunctions can be derived. In particular, neglecting the friction effects and considering a region of smooth flow we associate to (5) the following quasi-linear form

\[
\frac{\partial W}{\partial t} + A_1(W) \frac{\partial W}{\partial x} + A_2(W) \frac{\partial W}{\partial y} = 0, \quad (6)
\]
where \( \mathbf{W} = (u, v, h)^T \) and

\[
A_1(\mathbf{W}) = \begin{bmatrix}
  u & 0 & g \\
  0 & u & 0 \\
  h & 0 & u
\end{bmatrix}, \quad A_2(\mathbf{W}) = \begin{bmatrix}
  v & 0 & 0 \\
  0 & v & g \\
  0 & h & v
\end{bmatrix}.
\]

(7)

The characteristic equation associated with (6) is then given by

\[
\det(\mu I + cA_1 + sA_2) = 0,
\]

(8)

where \( I \) is the identity matrix. By choosing \( c \) and \( s \) such that \( c^2 + s^2 = 1 \), we compute immediately the eigenvalues of the system \( \lambda_1 = -(cu + sv), \lambda_{2,3} = -(cu + sv) \pm \sqrt{gh} \), together with the associated eigenfunctions

\[
w_1 = \begin{bmatrix}
  \sin(\phi) \\
  -\cos(\phi) \\
  0
\end{bmatrix}, \quad w_{2,3} = \begin{bmatrix}
  \pm \frac{g}{h} \cos(\phi) \\
  \pm \frac{g}{h} \sin(\phi) \\
  1
\end{bmatrix},
\]

where \( \phi \) is the direction of the characteristic lines. Finally, we refer to [1] for a stability analysis of the 2D shallow water equations in sub-critical flow regimes.

3 HETEROGENEOUS DOMAIN DECOMPOSITION ALGORITHM

We consider the decomposition of Figure 1, where \( \Omega_{2D} \) is the two-dimensional domain in which we want to solve the 2D shallow water system (5) and \( \Omega_{1D} \) the one-dimensional domain in which we consider the 1D system (1).

![Figure 1: A 2D domain with a 1D domain for a simple channel](image)

We denote with \( \gamma \) the matching region of the two models. Due to the different nature of the two domains, \( \gamma_- = (-L/2, L/2) \) is an interval, while \( \gamma_+ \) is a point. In the sub-domain \( \Omega_{1D} \), we solve the 1D unviscid shallow water model, thus yielding in \( \gamma_+ \) the physical quantities \( A_{1D}, Q_{1D} \) and \( h_{1D} \) and, consequently, \( u_{1D} = Q_{1D}/A_{1D} \). On the other hand, the
2D unviscid Saint-Venant equations are solved in $\Omega_{2D}$ with associated physical quantities $A_{2D}$, $Q_{2D}$ and $h_{2D}$ in $\gamma_-$. The matter consists of relating quantities of different dimension. Concerning the coupling 2D-1D, the reduction of the two-dimensional information to one-dimensional quantities is rather easy. We can, for instance, average the 2D terms along the cross-section $\gamma$. With this aim, let us introduce the mean velocity, the mean total water depth and the mean discharge

$$\bar{u}_{2D} = \frac{1}{L} \int_{-L/2}^{L/2} u(a, y) \, dy,$$
$$\bar{h}_{2D} = \frac{1}{L} \int_{-L/2}^{L/2} h(a, y) \, dy = \frac{A_{2D}}{L},$$
$$\bar{Q}_{2D} = A_{2D} \bar{u}_{2D},$$

where $A_{2D}$ is the area of the section at the matching point.

In order to choose the suitable coupling conditions, following [5] in the case of confined fluids, we demand the continuity of the following quantities in $\gamma$:

1. $A_{2D} = A_{1D}$, with $A_{1D} = h_{2D}L$;
2. $\bar{Q}_{2D} = Q_{1D}$;
3. $\sqrt{\bar{h}_{2D}g} + \bar{u}_{2D} = \sqrt{h_{1D}g} + \frac{Q_{1D}}{A_{1D}}$.

Notice that conditions 1. and 3. would suffice as 2. is automatically guaranteed when 1. and 3. hold.

We assume unidirectional flow, therefore we have a sub-critical outflow for the 2D system (with \textit{two outgoing characteristics}) and a sub-critical inflow for the one-dimensional problem with \textit{an incoming characteristic} (the super-critical situation is less interesting from the domain decomposition view point). According to the stability results, to choose which conditions have to be enforced to the 1D and the 2D model respectively, we adopt the following iterative strategy: at each time $t$, we solve the 2D and the 1D model with the following boundary conditions,

1. condition 1. is used for imposing the total depth at the outlet of the 2D model;
2. condition 3. is used at the inlet of the 1D model;

the iterative procedure ends when the difference between $h_{1D}$ and $\bar{h}_{2D}$ is less than a given tolerance. Usually few iterations (1 or 2) are sufficient to ensure an accurate solution at the interface and for this reason we are not adopting in the numerical results any sub-iteration strategy. Moreover at the inlet of the 2D model we impose the total depth $h(t)$ as function of time, whilst at the outlet of the 1D model a non-reflecting boundary condition is employed. Only a remark on the viscid case: in this case, due to the different nature of the two systems, we impose the continuity of the normal flux and the discharge.
4 NUMERICAL VALIDATION

For the numerical solution of the 2D model we use the finite element solver proposed originally in [12] for the 3D shallow water system. As for the 1D model a finite volume method has been employed.

To test the effectiveness of the proposed algorithm we consider a straight channel 320m long and 40m wide, divided into two equal parts of 160m long: in the first part we use the 2D model while in the second one the 1D model is solved. At the inlet of the 2D model we impose $h(t) = 0.5 \sin(2\pi t/10) + 5$ if $t \leq 5$, 5 otherwise: thus a traveling wave along the channel is generated. For the space discretization of both 1D and 2D models a mesh size $H = 1$ is used, while for the time discretization we choose a time-step $\Delta t = 0.05$.

In Figures 2 and 3 we show a comparison between the elevation computed with the full two-dimensional shallow water system and that computed with the 2D-1D coupling at two different times. The wave travels from the 2D to the 1D model without any significant distortion and a very small reflected wave is present in the 2D model. These results confirm the effectiveness of the proposed coupling algorithm.

As a second test case, we consider the river bifurcation of Figure 4. The one-dimensional system is solved only in the right channels after the bifurcation. We consider the same initial condition of the previous test case. The computed elevations (see Figures 5, 6) confirm again the effectiveness of the proposed algorithm.

5 CONCLUSIONS AND FUTURE DEVELOPMENTS

In this work we have proposed a coupling strategy between two 1D and 2D shallow water models and the preliminary numerical results in Section 4 confirm the soundness of our analysis.
Figure 3: The elevation at time $t = 250$ seconds using the full 2D shallow water system (left) and the 2D-1D coupling (right).

Figure 4: The computational domain

In the future, our goal will be to compare this a priori coupling strategy with an adaptive modeling technique driven by suitable a posteriori modeling error estimators. We will refer essentially to the recent theory provided in [3], [9], [10] and [11].
Figure 5: Computed elevation at $t = 250$ for the full two-dimensional model and the heterogeneous one

Figure 6: Computed elevation at $t = 300$ for the full two-dimensional model and the heterogeneous one

REFERENCES


