A Short Introduction to Signal Processing

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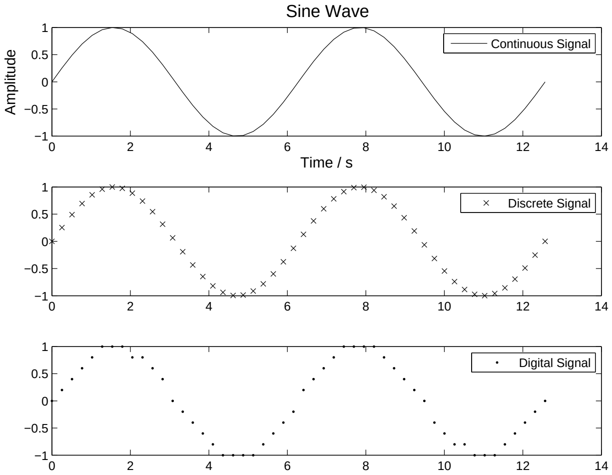
Definition of Signal

- A *signal* is a function of an independent variable such as time, distance, position, or temperature. Some examples of biomedical signals are:
  - Electrocardiogram (ECG), electroencephalogram (EEG) and magnetoencephalogram (MEG)
- A signal is said to be *continuous* when its domain is the set of real numbers, and *discrete* otherwise
- Discrete signals are presented as sequences of numbers called *samples*
- An *analog signal* is a real-valued continuous signal
- A *digital signal* is discrete in time and value

A Continuous, Discrete, and Digital Signal



Definition of Signal Processing

- *Signal processing* usually refers to:
  - Signal generation
  - Modifying signals
  - Extracting information from signals
- Signal processing benefits from improvements in the areas of electrical engineering, applied mathematics, statistics, mathematical information technology, ..

Definition of Frequency

- ”Frequency is the number of occurrences of a repeating event per unit time” –Wikipedia
- The SI unit of frequency is hertz (symbol Hz, 1 Hz = 1 / s)

Sampling, Sampling Rate and Sampling Theorem

- Sampling is the process of converting a continuous signal to a discrete one
- Sampling rate, usually denoted by f_s, is the number of samples per second collected from a continuous signal
  - Sampling rate is given in the unit of hertz
- "If a function x(t) contains no frequencies higher than B hertz, it is completely determined by giving its ordinates at a series of points spaced $\frac{1}{2B}$ seconds apart" –Shannon
  - For example, consider the human hearing sense. The human hearing range is about from 20 Hz to 20 kHz, so the sampling frequency of audio signals must be at least 40 kHz to include all audible frequencies (audio compact discs use 44.1 kHz, telephones 16 kHz)

Signal Processing Domains

- Signals are usually studied in
  - time-domain (with respect to time)
  - frequency-domain (with respect to frequency)
  - time and frequency domains simultaneously, using some time-frequency representation (TFR)
- Fourier transforms can be used to transform signals from time-domain to frequency-domain, and vice versa
- Time-frequency representations can be computed using short-time Fourier transform (STFT) or wavelets

Discrete Fourier Transform

- The frequency-domain representation of a digital time-domain signal $x[t]$ can be calculated using the discrete Fourier transform (referred as DFT or the *analysis equation*), defined as:

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-2\pi ik\frac{n}{N}} \tag{1}$$

$$= \sum_{n=0}^{N-1} x[n][cos(-2\pi k\frac{n}{N}) + isin(-2\pi k\frac{n}{N})] \tag{2}$$

where $0 \leq k \leq N - 1$ and i is the imaginary unit. The equation (2) is obtained through the Euler's formula

$$e^{i\theta} = cos(\theta) + isin(\theta) \tag{3}$$

Inverse Discrete Fourier Transform

- The time-domain representation of a frequency-domain signal $X[k]$ can be calculated using the inverse discrete Fourier transform (referred as IDFT or the *synthesis equation*), defined as:

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{-2\pi i k \frac{n}{N}} \tag{4}$$

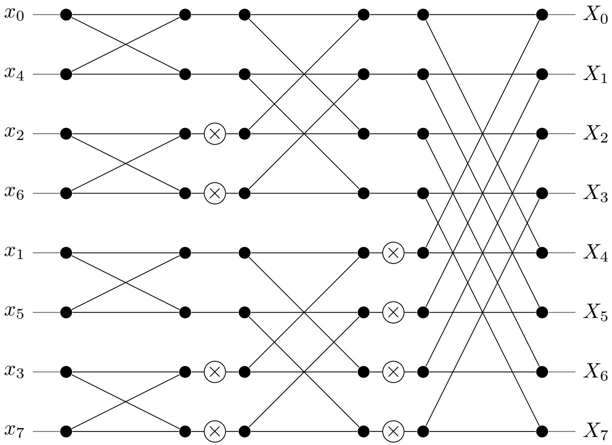
$$\tag{5}$$

where $0 \leq n \leq N - 1$ and i is the imaginary unit.

Fast Fourier Transform

- Fast Fourier transform (FFT) is ”the most important algorithm of our lifetime”. It is needed at least for:
  - Signal processing (convolution, digital filters)
  - Fast multiplication of large integers
  - Solving partial differential equations
  - Magnetic resonance imaging (MRI)
- The computational complexity of the discrete Fourier transform is $O(N^2)$ where N is the signal length
  - FFT produces the exact same result in $O(N \log N)$ operations

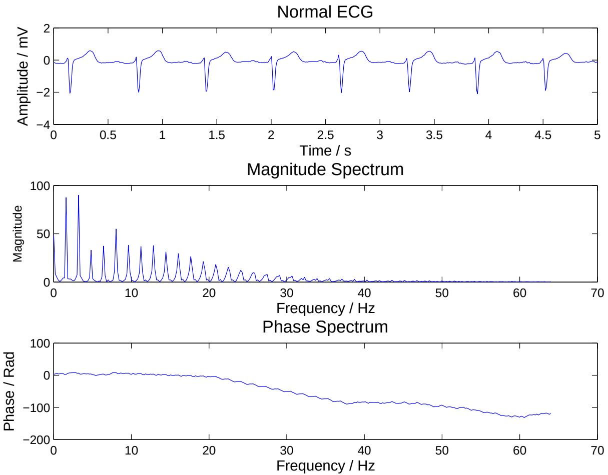
FFT Butterfly Diagram



Magnitude and Phase Spectrums

- DFT is complex valued $\rightarrow$ carries information about magnitude and phase
- The magnitude spectrum of a frequency-domain signal X is given by its absolute value $|X|$
- The phase spectrum of a frequency domain-signal X is given by it argument $arg(X)$

Magnitude and Phase Spectrums of Normal ECG



Convolution

- Discrete convolution of signals f and g is defined as:

$$(f * g)[n] = \sum_{m=-\infty}^{\infty} f[n] \cdot g[n-m] \tag{6}$$

$$= \sum_{m=-\infty}^{\infty} f[n-m] \cdot g[m] \tag{7}$$

- The resulting new signal is usually viewed as a modified or filtered version of one of the original signals

Convolution Theorem

- Convolution has an important property known as *convolution theorem*. Given two signals f and g the convolution theorem states that

$$\mathbb{F}\{f * g\} = \mathbb{F}\{f\} \cdot \mathbb{F}\{g\} \tag{8}$$

where $\mathbb{F}$ is used to denote DFT. By applying IDFT, denoted by $\mathbb{F}^{-1}$, on the both hand sides of the equation (6) one gets

$$f * g = \mathbb{F}^{-1}\{\mathbb{F}\{f\} \cdot \mathbb{F}\{g\}\} \tag{9}$$

so it is possible to calculate convolution efficiently by using Fourier tranforms.

Digital Filters

- Low-pass filters
  - Pass low frequencies, attenuate high frequencies
- High-pass filters
  - Pass high frequencies, attenuate low frequencies
- Band-pass filters
  - Pass frequencies within a specified range, attenuate frequencies outside that range
- Band-stop filters
  - Attenuate frequencies within a specified range, pass frequencies outside that range
- If a filter is linear and time-invariant, it is completely characterized and best described by its *frequency response*

Frequency Response

- If a filter is linear and time-invariant, its frequency response is given by either
  - $\mathbb{F}\{y\}/\mathbb{F}\{x\}$, where x is the input signal and y the output signal
  - By taking the Fourier transform of the *impulse response* of the filter, which is the filter's response to Kronecker's delta

$$\delta(n) = \begin{cases} 1, n = 0 \\ 0, n \neq 0 \end{cases}$$