ON THE NUMERICAL SOLUTION OF AXISYMMETRIC DOMAIN OPTIMIZATION PROBLEMS BY DUAL FINITE ELEMENT METHOD

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Abstract: An axisymmetric second order elliptic problem with mixed boundary conditions is considered. A part of the boundary has to be found so as to minimize a cost functional, which is given in terms of the cogradient of the solution. The numerical realization using the dual finite element method is presented in detail. Numerical examples are given.

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Introduction

One often meets elliptic problems in three-dimensional domains Ω which are generated by the rotation of a bounded plane domain D around an axis. Then the most suitable approach is to use cylindrical coordinates. If the data of the problem are axial symmetric, the problem is then reduced to the two-dimensional domain D.

In the paper [9] some axisymmetric domain optimization problems were solved numerically using the standard finite element method. In this paper we solve a similar problem numerically using the dual finite element method. The motivation for the use of the dual approach is that the cost function is given purely in terms of the cogradient of the solution. The dual approach allows us to calculate the cogradient directly. The formulation of the problem with proofs for the existence of a solution and convergence of approximations, was given by Hlaváček in [8].

In this work the modified definition of the set of admissible designs, presented in [9], is adopted. Using the techniques developed by Hlaváček and Křížek in [7] the implementation of the dual approach can be done in the similar way as in the case of the standard primal approach.

1. The state problem

We shall consider a class of admissible domains $D(\alpha)$, where

$$D(\alpha) = \{(r, z) \mid 0 < r < \alpha(z),\ 0 < z < 1\}$$

and the function $\alpha(z)$ – the design variable – belongs to the following set of admissible functions

$$U_{ad} = \{\alpha \in C^{(1),1}([0,1]) \,|0 < \alpha_{\min} \le \alpha(z) \le \alpha_{\max},\ |\alpha'(z)| \le C_1,$$
$$|\alpha''(z)| \le C_2 \text{ a.e. in } (0,1),\ \int_0^1 \alpha^2\, dz = C_3\}$$

with given positive constants $\alpha_{\min},\ \alpha_{\max},\ C_1,\ C_2,\ C_3$. Here $C^{(1),1}([0,1])$ denotes the space of Lipschitz functions with Lipschitz-continuous derivatives.

Let $\Gamma(\alpha)$ denote the graph of the function α,

$$\Gamma_0 = \partial D(\alpha) \cap \{r = 0\},\ \Gamma_1(\alpha) = \partial D(\alpha) \cap \{z = 0\},\ \Gamma_2(\alpha) = \partial D(\alpha) \cap \{z = 1\}.$$

FIGURE 1.1

We shall consider the following boundary value problem

$$\tag{1.1} \begin{cases} -\displaystyle\sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left(A_i(x)\frac{\partial u}{\partial x_i}\right) = \hat{f} \quad \text{in } \Omega(\alpha), \\ \displaystyle\sum_{i=1}^{3} A_i \frac{\partial u}{\partial x_i}\nu_i = 0 \quad \text{on } S_1(\alpha) \cup S_2(\alpha), \\ u = 0 \quad \text{on } S(\alpha), \end{cases}$$

where $\Omega(\alpha)$ is generated by rotation of $D(\alpha)$ around the x_3-axis, $S_i(\alpha)$ by rotation of $\Gamma_i(\alpha),\ i = 1,2$ and $S(\alpha)$ by rotation of $\Gamma(\alpha)$, (see Figure 1.1), ν_i are components of the unit outward normal with respect to $\partial\Omega(\alpha)$.

Let $\hat{\Omega}$ be the cylindrical domain generated by rotation of the rectangle $\hat{D} = (0,\delta) \times (0,1),\ \delta > \alpha_{\max}$.

Assume that the function $\hat{f}$ in (1.1) is determined as the restriction to $\Omega(\alpha)$ of an axisymmetric function $\hat{f} \in L^2(\hat{\Omega})$.

Assume that the coefficients A_i are restrictions to $\Omega(\alpha)$ of axisymmetric functions $A_i \in L^\infty(\hat{\Omega}),\ A_1 = A_2$ a.e. and a positive constant a_0 exists such that

$$\tag{1.2} A_i(x) \ge a_0 \quad \text{a.e. in } \hat{\Omega}.$$

Let us denote $A_1 = A_2 = a_r,\ A_3 = a_z,\ f(r,z) = \hat{f}(r\cos\vartheta, r\sin\vartheta, z)$.

Passing to the cylindrical coordinate system, we obtain the following state problem:

$$\tag{1.3} \begin{cases} -\dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r a_r \dfrac{\partial y}{\partial r}\right) - \dfrac{\partial}{\partial z}\left(a_z \dfrac{\partial y}{\partial z}\right) = f \quad \text{in } D(\alpha), \\ a_z \dfrac{\partial y}{\partial z} = 0 \quad \text{on } \Gamma_1(\alpha) \cup \Gamma_2(\alpha), \\ y = 0 \quad \text{on } \Gamma(\alpha). \end{cases}$$

Let $k \geq 0$ and n be integers. We shall denote by $L^m_{r^n}(D)$ the space of measurable functions u, for which

$$\|u\|^m_{0,r^n,D} = \int_D |u|^m r^n \, dr \, dz < +\infty, \quad m = 1, 2.$$

We shall denote by $W^k_{2,r^n}(D)$ the weighted Sobolev space with the weight r^n and the norm

$$\|u\|_{k,r^n,D} = \left(\sum_{|s| \leq k} \int_D |\mathrm{D}^s u|^2 \, r^n \, dr \, dz \right)^{\frac{1}{2}},$$

where D^s denotes any partial derivative of the order s. The same notations will be used also for vector functions.

The weak formulation of the state problem (1.3) reads

$$\text{(1.4)} \quad \begin{cases} \text{Find } y = y(\alpha) \in V(\alpha) \text{ such that} \\ \displaystyle\int_{D(\alpha)} \left(a_r \frac{\partial y}{\partial r} \frac{\partial v}{\partial r} + a_z \frac{\partial y}{\partial z} \frac{\partial v}{\partial z} \right) r \, dr \, dz = \int_{D(\alpha)} f v \, r \, dr \, dz \quad \forall v \in V(\alpha), \end{cases}$$

where

$$V(\alpha) = \{ v \in W^1_{2,r}(D(\alpha)) \mid \gamma v = 0 \text{ on } \Gamma(\alpha) \}.$$

The trace operator $\gamma : W^1_{2,r}(D(\alpha)) \to L^2_r(\Gamma(\alpha))$ is well defined also in the axisymmetric case – see [5].

Lemma 1.1. *The state problem (1.4) has a unique weak solution $y(\alpha)$ for all $\alpha \in U_{ad}$.*

Proof. There exists a positive constant c such that

$$\text{(1.5)} \quad \int_{D(\alpha)} |\operatorname{grad} u|^2 \, r \, dr \, dz \geq c \|u\|^2_{1,r,D(\alpha)}$$

holds for all $u \in V(\alpha)$ and $\alpha \in U_{ad}$. (For the proof, see [5]–Lemma 3). Using (1.2) and (1.5) we derive that the problem is $V(\alpha)$-elliptic and therefore uniquely solvable for any $\alpha \in U_{ad}$. Then $\operatorname{cograd} y(\alpha) \in \left[L^2_r(D(\alpha)) \right]^2$.

2. Setting of the domain optimization problem and dual variational formulation of the state problem

We define the *Domain Optimization Problem*:

$$(\mathbb{P}) \quad \begin{cases} \text{Find } \alpha^* \in U_{ad} \text{ such that} \\ J(\alpha^*, y(\alpha^*)) \leq J(\alpha, y(\alpha)) \quad \forall \alpha \in U_{ad}, \end{cases}$$

where $y(\alpha)$ denotes the solution of the state problem (1.4) and

(2.1)

$$J(\alpha, y(\alpha)) = \frac{1}{2} \int_{D(\alpha)} |\operatorname{cograd} y(\alpha) - \mathbf{g}|^2 \, r \, dr \, dz, \quad \mathbf{g} \in \left[L^2_r(\hat{D}) \right]^2,$$

$$\operatorname{cograd} y = \left(a_r \frac{\partial y}{\partial r}, \ a_z \frac{\partial y}{\partial z} \right)^{\mathrm{T}}.$$

Since the cost functional is given in terms of the cogradient of the solution, we shall employ the dual variational formulation of the state problem (1.4). Let us recall the latter formulation, the derivation of which can be found e.g. in the paper [7]–Part I, Section 2.

Let us introduce the notation $\mathbb{H}(\alpha) = \left[L_r^2(D(\alpha))\right]^2$ and the following bilinear form in $\mathbb{H}(\alpha) \times \mathbb{H}(\alpha)$

$$
\begin{aligned}
(\mathbf{q}, \mathbf{p})_{\mathbb{H}(\alpha)} &= \int_{D(\alpha)} (a_r^{-1} q_r p_r + a_z^{-1} q_z p_z)\, r\, dr\, dz, \\
\|\mathbf{q}\|_{\mathbb{H}(\alpha)} &= \sqrt{(\mathbf{q}, \mathbf{q})_{\mathbb{H}(\alpha)}}.
\end{aligned}
$$

It is easy to see that the norms $\|\cdot\|_{\mathbb{H}(\alpha)}$ and $\|\cdot\|_{0,r,D(\alpha)}$ are equivalent by virtue of (1.2). Moreover, let us define

$$
\begin{aligned}
B(\alpha; \mathbf{q}, v) &= \int_{D(\alpha)} \left(q_r \frac{\partial v}{\partial r} + q_z \frac{\partial v}{\partial z} \right) r\, dr\, dz, \\
L(v) &= \int_{D(\alpha)} f v\, r\, dr\, dz, \\
\mathcal{S}(\mathbf{q}) &= \frac{1}{2} \|\mathbf{q}\|^2_{\mathbb{H}(\alpha)}, \\
Q_f(\alpha) &= \{\mathbf{q} \in \mathbb{H}(\alpha) \mid B(\alpha; \mathbf{q}, v) = L(v) \quad \forall v \in V(\alpha)\}.
\end{aligned}
$$

The principle of minimum complementary energy then gives

$$
\mathbf{q}(\alpha) = \operatorname*{argmin}_{\mathbf{t} \in Q_f(\alpha)} \mathcal{S}(\mathbf{t}) \tag{2.2}
$$

if and only if $\mathbf{q}(\alpha) = \operatorname{cograd} y(\alpha)$.

We assume that

$$
\int_0^r t\, f(t, z)\, dt \in L^2_{1/r}(\hat{D}). \tag{2.3}
$$

Then the following vector field

$$
\boldsymbol{\lambda} = \left(-\frac{1}{r} \int_0^r t\, f(t, z)\, dt,\ 0 \right)^{\mathrm{T}} \tag{2.4}
$$

belongs to the set $Q_f(\alpha)$ for any $\alpha \in U^0_{ad}$, where

$$
U^0_{ad} = \{\alpha \in C^{(0),1}([0, 1]) \mid \alpha_{\min} \leq \alpha(z) \leq \alpha_{\max}\}.
$$

Defining the subspace

$$
Q(\alpha) = \{\mathbf{q} \in \mathbb{H}(\alpha) \mid B(\alpha; \mathbf{q}, v) = 0 \quad \forall v \in V(\alpha)\},
$$

we may write $Q_f(\alpha) = \boldsymbol{\lambda} + Q(\alpha)$. Substituting $\mathbf{q} = \boldsymbol{\lambda} + \mathbf{p},\ \mathbf{p} \in Q(\alpha)$ into (2.2), we obtain that

$$
\mathbf{p}(\alpha) = \operatorname*{argmin}_{\mathbf{t} \in Q(\alpha)} \{\mathcal{S}(\mathbf{t}) + (\boldsymbol{\lambda}, \mathbf{t})_{\mathbb{H}(\alpha)}\} \tag{2.5}
$$

if and only if $\mathbf{p}(\alpha) = \operatorname{cograd} y(\alpha) - \boldsymbol{\lambda}$. The sufficient and necessary condition for the minimizer $\mathbf{p}(\alpha) \in Q(\alpha)$ is

$$(\mathbf{p}(\alpha), \mathbf{t})_{\mathbb{H}(\alpha)} = -(\boldsymbol{\lambda}, \mathbf{t})_{\mathbb{H}(\alpha)} \quad \forall \mathbf{t} \in Q(\alpha). \tag{2.6}$$

The latter minimum problem has a unique solution $\mathbf{p}(\alpha)$ for any $\alpha \in U_{ad}$.

The Domain Optimization Problem $(\mathbb{P})$ is replaced by the following Equivalent Domain Optimization Problem:

$$(\mathbb{P}^*) \qquad \begin{cases} \text{Find } \alpha^* \in U_{ad} \text{ such that} \\ J^*(\alpha^*, \mathbf{q}(\alpha^*)) \leq J^*(\alpha, \mathbf{q}(\alpha)) \quad \forall \alpha \in U_{ad}, \end{cases}$$

where

$$J^*(\alpha, \mathbf{q}) = \frac{1}{2} \int_{D(\alpha)} |\mathbf{q} - \mathbf{g}|^2 \, r \, dr \, dz, \quad \mathbf{q}(\alpha) = \boldsymbol{\lambda} + \mathbf{p}(\alpha), \tag{2.7}$$

$\boldsymbol{\lambda}$ is defined by the formula (2.4) and $\mathbf{p}(\alpha)$ is the solution of (2.6).

3. Approximation by finite elements

In the present Section we propose an approximate solution of the domain optimization problem $(\mathbb{P}^*)$, making use of piecewise linear design variable and quadrilateral finite elements with bilinear shape functions for solving the state problem.

Let N be a positive integer and $h = \frac{1}{N}$. We denote by Δ_j, $j = 1, 2, ..., N$, the subintervals $[(j-1)h, jh]$ and introduce the set

$$\begin{aligned} U_{ad}^h = \{\alpha_h \in C^{(0),1}([0,1]) \, | 0 < \alpha_{\min} \leq \alpha_h(z) \leq \alpha_{\max}, \\ \alpha_h|_{\Delta_j} \in P_1(\Delta_j) \ \forall \Delta_j, \ j = 1, 2, ..., N, \\ |\alpha_h'| \leq C_1, \ |\delta_h^2 \alpha_h| \leq C_2, \ \int_0^1 \alpha_h^2 \, dz = C_3\}, \end{aligned}$$

where $C^{(0),1}([0,1])$ denotes the set of Lipschitz-functions, $P_1(\Delta_j)$ is the set of linear functions defined on Δ_j and $\delta_h^2 \alpha_h$ denotes the second difference

$$\delta_h^2 \alpha_h(jh) = \frac{1}{h^2} \big[\alpha_h((j+1)h) - 2\alpha_h(jh) + \alpha_h((j-1)h)\big], \quad j = 1, ..., N-1. \tag{3.1}$$

Let $D_h = D(\alpha_h)$ denote the domain bounded by the graph $\Gamma_h = \Gamma(\alpha_h)$ of the function $\alpha_h \in U_{ad}^h$. The polygonal domain D_h will be carved into quadrilaterals K in the following way. We choose $\alpha_0 \in (0, \alpha_{\min})$ and introduce a uniform mesh on the rectangle $\mathcal{R} = [0, \alpha_0] \times [0, 1]$, independent of α_h, if h is fixed.

In the remaining part $D_h \backslash \mathcal{R}$ let the nodal points divide the segments $[\alpha_0, \alpha_h(jh)]$, $j = 1, 2, ..., N$, into N' equal segments, where

$$N' = 1 + \lfloor (\alpha_{\max} - \alpha_0) N \rfloor$$

and the brackets $\lfloor \ \rfloor$ denote the integer part of the number inside. Consequently, one obtains a regular family $\{\mathcal{T}_h(\alpha_h)\}$, $h \to 0$, $\alpha_h \in U_{ad}^h$, of meshes (cf. [1]).

In what follows, we shall assume in addition that

$$f(\cdot, z)|_{\hat{D} \backslash \mathcal{R}} \in C([\alpha_0, \delta]), \ a_r(\cdot, z)|_{\hat{D} \backslash \mathcal{R}}, \ a_z(\cdot, z)|_{\hat{D} \backslash \mathcal{R}} \in C^1([\alpha_0, \delta]) \tag{3.2}$$

for almost all $z \in (0,1)$.

We define the approximate state problem

$$\begin{cases} \text{Find } \mathbf{p}^h(\alpha_h) \in S_h \text{ such that} \\ (\mathbf{p}^h(\alpha_h), \mathbf{t}^h)_{\mathbb{H}(\alpha_h)} = -(\boldsymbol{\lambda}, \mathbf{t}^h)_{\mathbb{H}(\alpha_h)} \quad \forall \mathbf{t}^h \in S_h, \end{cases} \tag{3.3}$$

where $S_h \subset Q(\alpha)$ is a finite dimensional subspace.

The equilibrium condition in the definition of $Q(\alpha)$ implies that standard finite elements cannot be directly utilized. Therefore we use the method proposed in the paper [7].

We introduce the spaces

$$X(\alpha) = W^1_{2,r}(D(\alpha)) \cap L^2_{1/r}(D(\alpha))$$

and

$$Y(\alpha) = W^1_{2,r^3}(D(\alpha)) \cap L^2_r(D(\alpha))$$

with the norms

$$\|\varphi\|_{X(\alpha)} = \left(\int_{D(\alpha)} \left(\left(\frac{\varphi}{r}\right)^2 + |\operatorname{grad}\varphi|^2 \right) r\,dr\,dz \right)^{\frac{1}{2}}$$

and

$$\|v\|_{Y(\alpha)} = \left(\int_{D(\alpha)} \left(v^2 + |\operatorname{grad} v|^2\, r^2 \right) r\,dr\,dz \right)^{\frac{1}{2}}$$

respectively. The operator

$$\operatorname{curl}\varphi = \left(\frac{\partial\varphi}{\partial z},\ -\frac{\varphi}{r} - \frac{\partial\varphi}{\partial r} \right)^{\mathrm{T}} \tag{3.4}$$

is then well-defined on the subspace

$$W(\alpha) = \{\varphi \in X(\alpha) \mid \gamma\varphi = 0 \text{ on } \Gamma_1(\alpha) \cup \Gamma_2(\alpha)\}. \tag{3.5}$$

For any $\alpha \in U^0_{ad}$ the space $Q(\alpha)$ can be identified with

$$\operatorname{curl} W(\alpha) = \{\mathbf{q} \in \mathbb{H}(\alpha) \mid \exists\varphi \in W(\alpha) \text{ such that } \mathbf{q} = \operatorname{curl}\varphi\} \tag{3.6}$$

as $\operatorname{curl} : W(\alpha) \to Q(\alpha)$ is a one-to-one mapping ([7]–Theorem 4.6). For any $\alpha \in U^0_{ad}$ and any $u \in X(\alpha)$ we have the inequalities

$$\frac{1}{\sqrt{3}}\|u\|_{X(\alpha)} \le \|\frac{u}{r}\|_{Y(\alpha)} \le \sqrt{3}\|u\|_{X(\alpha)}.$$

Thus, if we construct approximations of $\mathbf{q} \in Q(\alpha)$, we may write

$$\mathbf{q} = \operatorname{curl}\varphi = \operatorname{curl}(r\psi), \quad \psi \in Y_0(\alpha) = \{v \in Y(\alpha) \mid v = 0 \text{ on } \Gamma_1(\alpha) \cup \Gamma_2(\alpha)\} \tag{3.7}$$

and approximate the function ψ.

Let us define the finite element space

$$\Sigma_h = \{u \in C(\overline{D(\alpha_h)}) \mid u|_K \circ F_K \in Q_1(\hat{K}) \quad \forall K \in \mathcal{T}_h(\alpha_h)\}, \tag{3.8}$$

where $Q_1(\hat{K})$ denotes the space of bilinear polynomials defined on $\hat{K}$ and $F_K \in [Q_1(\hat{K})]^2,\ F_K : \hat{K} \to K,\ \hat{K} = [-1,1] \times [-1,1]$.

We shall construct subspaces $S_h \subset Q(\alpha_h)$. Let us define the set

$$Y_h = \{u_h \mid u_h = rw_h,\ w_h \in \Sigma_h,\ w_h = 0 \text{ on } \Gamma_1(\alpha_h) \cup \Gamma_2(\alpha_h)\} \tag{3.9}$$

and

$$S_h = \operatorname{curl} Y_h. \tag{3.10}$$

It is easy to verify that $Y_h \subset W(\alpha_h)$ and then $S_h \subset Q(\alpha_h)$ follows.

$$\mathbf{q}^h \in S_h \implies \mathbf{q}^h = \left(r\,\frac{\partial w_h}{\partial z},\ -2w_h - r\,\frac{\partial w_h}{\partial r} \right)^{\mathrm{T}}, \quad w_h \in \Sigma_h. \tag{3.11}$$

Moreover, the normal component $\mathbf{q}^h \cdot \nu$ is continuous across element boundaries.

From the bijectivity of the operator curl and the definition (3.10) it follows that $\dim S_h = \dim Y_h$. Moreover if $\{\psi_i\}_{i=1}^M$ is the basis of Σ_h then

$$\{\Phi_i\}_{i=1}^M,\ \Phi_i = \operatorname{curl}(r\psi_i) \tag{3.12}$$

is the basis of S_h.

Seeking $\mathbf{p}^h$ as a linear combination of the basis functions

$$\mathbf{p}^h = \sum_{i=1}^M c_i \Phi_i,$$

and defining the approximate solution of the state problem by

$$(\mathbf{p}^h, \mathbf{t})_{\mathbb{H}(\alpha_h)} = -(\boldsymbol{\lambda}, \mathbf{t})_{\mathbb{H}(\alpha_h)} \quad \forall \mathbf{t} \in S_h \tag{3.13}$$

we get for the unknowns c_i a linear system of equations

$$\sum_{i=1}^M c_i (\Phi_i, \Phi_j)_{\mathbb{H}(\alpha_h)} = -(\boldsymbol{\lambda}, \Phi_j)_{\mathbb{H}(\alpha_h)},\ j = 1, ..., M. \tag{3.14}$$

Taking into account (3.12), we have

$$\begin{aligned}(\Phi_i, \Phi_j)_{\mathbb{H}(\alpha_h)} &= (\operatorname{curl}(r\psi_i), \operatorname{curl}(r\psi_j))_{\mathbb{H}(\alpha_h)} \\ &= \int_{D(\alpha_h)} \left[a_r^{-1} r\,\frac{\partial \psi_i}{\partial z}\, r\,\frac{\partial \psi_j}{\partial z} + a_z^{-1} \left(2\psi_i + r\frac{\partial \psi_i}{\partial r} \right) \left(2\psi_j + r\frac{\partial \psi_j}{\partial r} \right) \right] r\,dr\,dz \equiv k_{ij},\end{aligned}$$

$$\begin{aligned}(\boldsymbol{\lambda}, \Phi_j)_{\mathbb{H}(\alpha_h)} &= (\boldsymbol{\lambda}, \operatorname{curl}(r\psi_j))_{\mathbb{H}(\alpha_h)} \\ &= \int_{D(\alpha_h)} \left[a_r^{-1} \lambda_r r\,\frac{\partial \psi_j}{\partial z} + a_z^{-1} \lambda_z \left(-2\psi_j - r\frac{\partial \psi_j}{\partial r} \right) \right] r\,dr\,dz \equiv f_j,\end{aligned}$$

which can be written in compact matrix form

$$\mathbf{K}\,\mathbf{c} = \mathbf{f}. \tag{3.15}$$

Proposition 3.1. *Let $\{\alpha_h\}$, $h \to 0$ be a sequence of $\alpha_h \in U_{ad}^h$, converging to a function α in $C([0,1])$. Then*

$$\mathbf{p}^{0h}(\alpha_h) \to \mathbf{p}^0(\alpha) \quad in \quad \left[L_r^2(\hat{D})\right]^2 \quad for\ h \to 0, \tag{3.16}$$

where $\mathbf{p}^{0h}(\alpha_h)$ is the solution of (3.13), extended by zero to the domain $\hat{D} \setminus \overline{D(\alpha_h)}$ and $\mathbf{p}^0(\alpha)$ is the solution of (2.6) extended by zero to $\hat{D} \setminus \overline{D(\alpha)}$.

Proof. One can use the same line of thought as in the proof of Proposition 1 of [8]. The only change consists in replacing the triangular finite elements by the isoparametric ones, defined in (3.8). Hence we also have to derive an error estimate for the latter.

We use partitions $\mathcal{T}_h(\delta)$ of the rectangle $\hat{D}$ into quadrilaterals, which create a regular family of extensions of $\mathcal{T}_h(\alpha_h)$, $h \to 0$, $\alpha_h \in U_{ad}^h$, the corresponding spaces $\Sigma_h(\delta)$ (extensions of (3.8)) and the interpolation operator

$$\Pi_h : C(\bar{\hat{D}}) \to \Sigma_h(\delta)$$
$$\Pi_h u(a_j) = u(a_j) \quad \text{at all nodal points } a_j \in \mathcal{T}_h(\delta).$$

We shall derive that

$$\|u - \Pi_h u\|_{Y(\delta)} \le ch\|u\|_{2,\hat{D}} \tag{3.17}$$

holds for all $u \in H^2(\hat{D})$. In fact, we may write for any quadrilateral $K \in \mathcal{T}_h(\delta)$

$$|u - \Pi_h u|_{m,r^n,K} \le \delta^{n/2}|u - \Pi_h u|_{m,K} \le c\delta^{n/2}h_K^{2-m}\|u\|_{2,K},$$

where $h_K = \operatorname{diam} K \le ch$, $m = 0, 1$ and n any positive integer (cf. [3]–p. 247 for the inequality). Consequently, we have

$$\begin{aligned}\|u - \Pi_h u\|_{Y(\delta)}^2 &= \sum_{K \in \mathcal{T}_h(\delta)} \left(|u - \Pi_h u|_{1,r^3,K}^2 + \|u - \Pi_h u\|_{0,r,K}^2\right) \\ &\le \sum_{K \in \mathcal{T}_h(\delta)} \hat{c}(h_K^2 + h_K^4)\|u\|_{2,K}^2 \le ch^2\|u\|_{2,\hat{D}}^2\end{aligned}$$

and (3.17) follows.

The rest of the proof is the same as in the above-mentioned proof in [8]. Q.E.D.

For a fixed parameter h, we define the Approximate Domain Optimization Problem:

$$(\mathbb{P}_h^*) \qquad \begin{cases} \text{Find } \alpha_h^* \in U_{ad}^h \text{ such that} \\ J^*(\alpha_h^*, \mathbf{q}^h(\alpha_h^*)) \le J^*(\alpha_h, \mathbf{q}^h(\alpha_h)) \quad \forall \alpha_h \in U_{ad}^h, \end{cases}$$

where $\mathbf{q}^h(\alpha_h) = \boldsymbol{\lambda} + \mathbf{p}^h(\alpha_h)$ and $\mathbf{p}^h(\alpha_h)$ is the solution of (3.13).

Proposition 3.2. *Let $\{\alpha_h\}$, $h \to 0$ be a sequence of $\alpha_h \in U_{ad}^h$, converging to a function α in $C([0,1])$. Then*

$$\lim_{h \to 0} J^*(\alpha_h, \mathbf{q}^h(\alpha_h)) = J^*(\alpha, \mathbf{q}(\alpha)). \tag{3.18}$$

Proof. Let us denote $\mathbf{q}^{0h} = \boldsymbol{\lambda} + \mathbf{p}^{0h}(\alpha_h)$, $\mathbf{q}^0 = \boldsymbol{\lambda} + \mathbf{p}^0(\alpha)$. Then we have by virtue of Proposition 3.1

$$\begin{aligned} J^*(\alpha_h, \mathbf{q}^h(\alpha_h)) &= \frac{1}{2}\int_{\hat{D}} |\mathbf{q}^{0h} - \mathbf{g}|^2\, r\, dr\, dz - \frac{1}{2}\int_{\hat{D}\setminus D(\alpha_h)} |\boldsymbol{\lambda} - \mathbf{g}|^2\, r\, dr\, dz \\ &\to \frac{1}{2}\int_{\hat{D}} |\mathbf{q}^0 - \mathbf{g}|^2\, r\, dr\, dz - \frac{1}{2}\int_{\hat{D}\setminus D(\alpha)} |\boldsymbol{\lambda} - \mathbf{g}|^2\, r\, dr\, dz \\ &= \frac{1}{2}\int_{D(\alpha)} |\mathbf{q}(\alpha) - \mathbf{g}|^2\, r\, dr\, dz = J^*(\alpha, \mathbf{q}(\alpha)) \end{aligned}$$

Q.E.D.

Proposition 3.3. *The problem* $(\mathbb{P}_h^*)$ *has at least one solution for any* $h = 1/N$.

Proof. The function $\alpha_h \in U_{ad}^h$ is completely determined by its nodal values $a_j = \alpha_h(jh),\ j = 0, 1, ..., N$. Thus $\alpha_h \in U_{ad}^h$ if and only if the vector $\mathbf{a} = (a_0, a_1, ..., a_N) \in A$, where A is a compact subset of $\mathbb{R}^{N+1}$.

By the construction of the finite element mesh $\mathcal{T}_h(\alpha_h)$ and by (3.2) the matrix $\mathbf{K}$ and the vector $\mathbf{f}$ in (3.15) depend continuously on the vector $\mathbf{a}$. Also the solution vector $\mathbf{c}$, which uniquely determines $\mathbf{p}(\alpha_h)$, depends continuously on $\mathbf{c}$. Then for extensions we have

$$\beta_h \to \alpha_h \quad \text{in } C([0,1]) \implies \mathbf{p}^{0h}(\beta_h) \to \mathbf{p}^{0h}(\alpha_h) \quad \text{in } \mathbb{H}(\delta).$$

Moreover, it is easy to see that

$$j^*(\mathbf{a}) = J^*(\alpha_h, \boldsymbol{\lambda} + \mathbf{p}^h(\alpha_h))$$

depends continuously on $\mathbf{a}$. Consequently, the minimum is attained in the set A.

Q.E.D.

Theorem 3.1. *Let* $\{\alpha_h^*\},\ h \to 0$, *be a sequence of solutions of problem* $(\mathbb{P}_h^*)$. *Then a subsequence* $\{\alpha_{\hat{h}}^*\}$ *exists, such that*

$$\alpha_{\hat{h}}^* \to \alpha^* \qquad \textit{in } C([0,1]),$$

where α^* *is a solution of the problem* $(\mathbb{P}^*)$.

The approximate state solutions $\mathbf{q}^{\hat{h}}(\alpha_{\hat{h}}^*)$ *converge in accordance with Proposition 3.1 to the solution* $\mathbf{q}(\alpha^*)$. *Any uniformly convergent subsequence of* $\{\alpha_h^*\}$ *has the properties mentioned above.*

Proof. Let $\beta \in U_{ad}$ be arbitrarily chosen. There exists a sequence $\{\beta_h\},\ h \to 0,\ \beta_h \in U_{ad}^h$, such that $\beta_h \to \beta$ in $C([0,1])$ (for the proof – see [9]–Lemma 3.1 and the Appendix in [5]).

Let us consider the set

$$\begin{gathered} U_{ad}^* = \{\alpha \in C^{(0),1}([0,1]) \mid 0 < \alpha_{\min} \le \alpha(x) \le \alpha_{\max},\ |\alpha'(z)| \le C_1, \\ \int_0^1 \alpha^2\, dz = C_3\}. \end{gathered}$$

Since $U_{ad}^h \subset U_{ad}^*\ \forall h$ and U_{ad}^* is compact in $C([0,1])$ (cf. Arzelà-Ascoli Theorem), there exists a subsequence $\{\alpha_{\hat{h}}^*\} \subset \{\alpha_h^*\}$, such that $\alpha_{\hat{h}}^* \to \alpha^*$ in $C([0,1])$. Using Lemma 3.2 of [9], we obtain that $\alpha^* \in U_{ad}$.

By definition we have

$$J^*(\alpha^*_{\hat{h}}, \mathbf{q}^{\hat{h}}(\alpha^*_{\hat{h}})) \le J^*(\beta_{\hat{h}}, \mathbf{q}^{\hat{h}}(\beta_{\hat{h}})) \quad \forall \hat{h}.$$

Letting $\hat{h} \to 0$ and using Proposition 3.2 on both sides, we obtain

$$J^*(\alpha^*, \mathbf{q}(\alpha^*)) \le J^*(\beta, \mathbf{q}(\beta)).$$

Consequently, α^* is a solution of $(\mathbb{P}^*)$, which is equivalent with $(\mathbb{P})$. The rest of the Theorem follows from Proposition 3.1.

Corollary 3.1. *There exists at least one solution of the problem* $(\mathbb{P})$.

Proof. follows from Proposition 3.3 and Theorem 3.1 .

4. Numerical realization

As $\mathbf{q}^h = \mathbf{p}^h + \boldsymbol{\lambda} = \sum_{i=1}^M c_i \Phi_i + \boldsymbol{\lambda} = \mathbf{Bc} + \boldsymbol{\lambda}$, the cost functional $J^*(\alpha_h, \mathbf{q}^h(\alpha_h))$ can be written in matrix form

$$\begin{aligned} J^*(\alpha_j, \mathbf{q}^h(\alpha_h)) &= \frac{1}{2} \sum_{K \in \mathcal{T}_h(\alpha_h)} \int_K |\mathbf{Bc} + \boldsymbol{\lambda} - \mathbf{g}|^2 \, r \, dr \, dz \\ &= \frac{1}{2} \sum_{K \in \mathcal{T}_h(\alpha_h)} \int_{\hat{K}} |\mathbf{Bc} + \boldsymbol{\lambda} - \mathbf{g}|^2 \, r \, |\mathbf{J}_K| \, d\rho \, d\zeta. \end{aligned} \tag{4.1}$$

Here $\hat{K}$ denotes some fixed reference element ($[-1,1] \times [-1,1]$, for example) and $|\mathbf{J}_K|$ denotes the Jacobian determinant of the coordinate transformation $F_K : \hat{K} \to K$.

The approximate domain optimization problem $(\mathbb{P}^*_h)$ is equivalent to the following nonlinear programming problem:

$$\begin{cases} \text{Find } \mathbf{a}^* \in A \text{ such that} \\ j^*(\mathbf{a}^*, \mathbf{c}(\mathbf{a}^*)) \le j^*(\mathbf{a}, \mathbf{c}(\mathbf{a})) \quad \forall \mathbf{a} \in A, \end{cases} \tag{4.2}$$

where

$$\begin{aligned} A = \{\mathbf{a} \in \mathbb{R}^{N+1} \, | & \alpha_{\min} \le a_i \le \alpha_{\max}, \ i = 0, 1, ..., N; \\ & - C_1 h \le a_i - a_{i-1} \le C_1 h, \ i = 1, 2, ..., N; \\ & - C_2 h^2 \le a_{i+1} - 2a_i + a_{i-1} \le C_2 h^2, \ i = 1, 2, ..., N-1; \\ & \frac{h}{2}(a_0^2 + a_N^2) + h \sum_{i=1}^{N-1} a_i^2 = C_3 \} \end{aligned}$$

The problem (4.2) is nonlinearly constrained. To solve it efficiently on a computer, one must calculate analytic gradients of the cost functional. The following result is standard:

Lemma 4.1. *Partial derivatives* $\partial j^*(\mathbf{a})/\partial a_j$ *are given by*

$$\frac{\partial j^*(\mathbf{a})}{\partial a_j} = \frac{\partial j^*(\mathbf{a}, \mathbf{c})}{\partial a_j} + \mathbf{h}^{\mathrm{T}} \left[\frac{\partial \mathbf{f}}{\partial a_j} - \left(\frac{\partial \mathbf{K}}{\partial a_j} \right) \mathbf{c} \right], \tag{4.3}$$

where $\mathbf{h}$ *is the solution of the adjoint equation*

$$\mathbf{K}\,\mathbf{h} = \nabla_{\mathbf{c}} j^*(\mathbf{a},\mathbf{c}). \tag{4.4}$$

Proof. See [9], for example.

In our case

$$\nabla_{\mathbf{c}} j^*(\mathbf{a},\mathbf{c}) = \sum_K \int_{\hat{K}} \mathbf{B}^{\mathrm{T}}(\mathbf{Bc} + \boldsymbol{\lambda} - \mathbf{g})\, r|\mathbf{J}_K|\, d\rho\, d\zeta \tag{4.5}$$

and

$$\begin{aligned} \frac{\partial j^*(\mathbf{a},\mathbf{c})}{\partial a_j} = &\sum_{\substack{K\in\mathcal{T}_h(\alpha_h)\\ K\cap\mathcal{R}=\emptyset}} \int_{\hat{K}} \Big[(\mathbf{B}')^{\mathrm{T}}(\mathbf{Bc} + \boldsymbol{\lambda} - \mathbf{g})\, r\, |\mathbf{J}_K| \\ &+ \mathbf{B}^{\mathrm{T}}(\mathbf{B}'\mathbf{c} + \frac{\partial \lambda_r}{\partial r} r' - \frac{\partial g_r}{\partial r} r' - \frac{\partial g_z}{\partial r} r')\, r\, |\mathbf{J}_K| \\ &+ \mathbf{B}^{\mathrm{T}}(\mathbf{Bc} + \boldsymbol{\lambda} - \mathbf{g})\, r'\, |\mathbf{J}_K| + \mathbf{B}^{\mathrm{T}}(\mathbf{Bc} + \boldsymbol{\lambda} - \mathbf{g})\, r\, |\mathbf{J}_K|' \Big]\, d\rho\, d\zeta. \end{aligned} \tag{4.6}$$

Above we have denoted $(\ )' = \partial(\ )/\partial a_j$. The terms $\partial\mathbf{f}/\partial a_j$, $\partial\mathbf{K}/\partial a_j$ in (4.3) and the terms $\mathbf{B}'$, r', $|\mathbf{J}_K|'$ in (4.6) can be calculated using techniques described in [9] and [10], for example.

5. Numerical examples

In this section we present numerical results of several test cases. In optimization we have used Sequential Quadratic Programming (SQP) algorithm E04VCF from the NAG-library. E04VCF is essentially the code NPSOL due to Gill et al. (see [4]). The state problem (3.15) and the adjoint problem (4.4) were solved iteratively using the Jacobi-conjugate gradient method. All computations were done in double precision using a HP 9000/370-workstation.

Example 5.1. In this example we have $f = -1$, $a_r = a_z = 1$, $\alpha_{\min} = 0.8$, $\alpha_{\max} = 1.2$, $C_1 = 2$, $C_2 = 10$, $C_3 = 1$, $\mathbf{g} = (r/2, 0)^{\mathrm{T}}$ and $h = 1/8$. In this case the optimal solution is known to be $\alpha^* \equiv 1$. As an initial quess we choose $\alpha_h^{(0)} \in U_{ad}^h$, with nodal values

$$\mathbf{a}^{(0)} = (0.8,\ 0.8871,\ 1.007,\ 1.1163,\ 1.1508,\ 1.1103,\ 0.995,\ 0.8751,\ 0.8)^{\mathrm{T}}.$$

The initial cost is $J^*(\alpha_h^{(0)}) = 7.61 \times 10^{-3}$. After 13 SQP-iterations and 72 CPU-seconds we obtained $\alpha_h^{(13)}$, for which

$$\begin{aligned} &J^*(\alpha_h^{(13)}) = 2.47 \times 10^{-11} \\ &\|\alpha^* - \alpha_h^{(13)}\|_\infty = 0.019\ . \end{aligned}$$

FIGURE 5.1

Example 5.2. Let $\alpha_{\min} = 0.8,\ \alpha_{\max} = 1.2,\ C_1 = 2,\ C_2 = 8,\ C_3 = 1,\ f = 1,\ a_z = 1$,

$$a_r = \begin{cases} \frac{1}{2}, & \text{for } z < \frac{1}{2} \\ 1, & \text{for } z > \frac{1}{2} \end{cases} \quad \text{and} \quad \mathbf{g} = \begin{cases} (r/4, -r/4)^{\mathrm{T}}, & \text{for } z < \dfrac{1}{2} \\ (r/4, -r/4)^{\mathrm{T}}, & \text{for } z > \dfrac{1}{2}. \end{cases}$$

We solved the problem with three different h:s. In all cases $\alpha_h^{(0)} \equiv 1$ was chosen as an initial quess.

In the case $h = 1/10$ after 8 SQP-iterations and 42 CPU-seconds we obtained $\alpha_{1/10}^{(8)}$ with $J^*(\alpha_{1/8}^{(8)}) = 9.44 \times 10^{-3}$. The initial cost was 18.7×10^{-3}. Plots of cogradient fields in the initial and final domains are shown in Figure 5.1.

In the case $h = 1/20$ after 14 SQP-iterations and 480 CPU-seconds we obtained $\alpha_{1/20}^{(14)}$ with a cost 9.32×10^{-3}. In Figure 5.2 the finite element mesh of the final domain is shown.

FIGURE 5.2

In the final case $h = 1/40$ after 17 SQP-iterations and 46 CPU-minutes we obtained $\alpha_{1/40}^{(17)}$ with a cost 9.26×10^{-3}. In Figure 5.3 the finite element mesh of the final domain is shown.

FIGURE 5.3

It is to be noted that in this example even with a coarse mesh fairly good approximations for the (possibly local) optimum α^* were obtained as

$$\frac{\max_{j=0,1,\ldots,10} \left|\alpha_{1/10}^{(13)}(j \cdot \frac{1}{10}) - \alpha_{1/40}^{(17)}(j \cdot \frac{1}{10})\right|}{\max_{j=0,1,\ldots,10} \left|\alpha_{1/10}^{(17)}(j \cdot \frac{1}{10})\right|} \approx 3\%.$$

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Figures

Figure 1.1

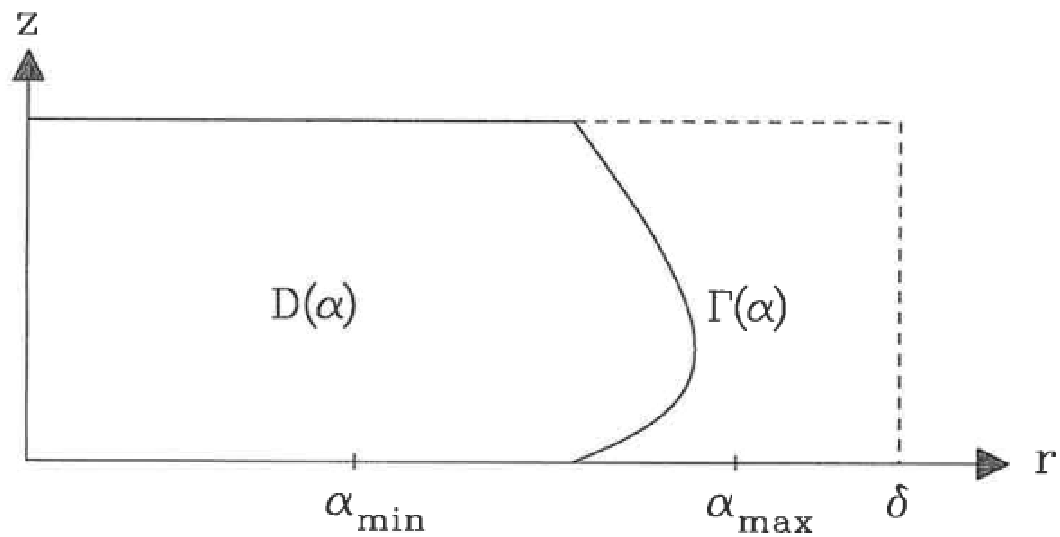
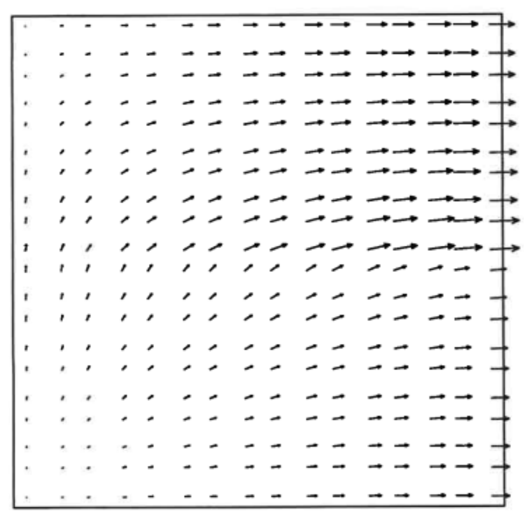


Figure 5.1



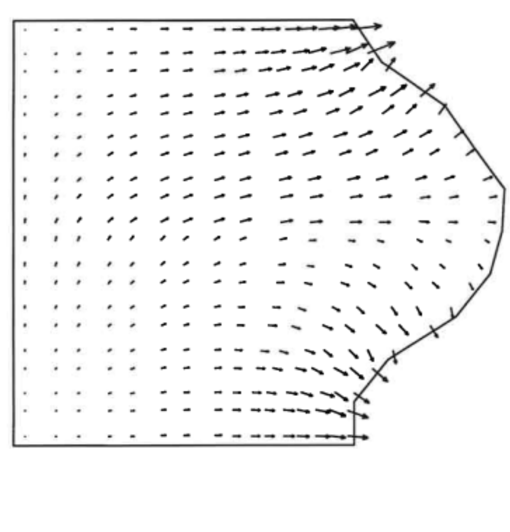


Figure 5.2

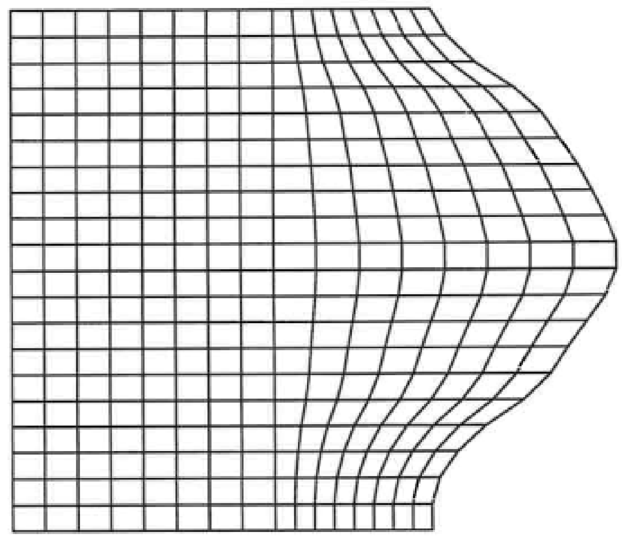


Figure 5.3