A FIXED DOMAIN APPROACH IN AN OPTIMAL SHAPE DESIGN PROBLEM

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A fixed domain approach in an optimal shape design problem

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Abstract. A fixed domain approach is presented for solving optimal shape design problems. In the proposed method the original optimal shape design problem is converted to a control problem settled in a fixed domain. The method is demonstrated in solving an optimal shape design problem arising from transmission problems. Results of numerical tests are presented.

Keywords. Optimal shape design, control approach.

Subject classification: 49E99, 35J67

1. INTRODUCTION

The standard way to solve optimal shape design problems numerically is the boundary variation method. In that method the unknown boundary is parametrized using a set of design parameters. The choice of this parametrization at the beginning of the optimization determines a restricted class of the domains that can be achieved. For example, if a single connected domain is assumed at the beginning, the optimal shape that is obtained, if it exists, is within this class of domains, although the true optimal domain may be doubly connected.

The implementation of the boundary variation method on computer is usually not a trivial task. Namely, at each step of the iterative optimization process, a new finite element mesh must be generated, Haslinger and Neittaanmäki [4].

From these points of view, fixed domain approach in optimal design problems are very useful and we quote the mapping method, Murat and Simon [7], the penalization method, Kawarada [5], the controllability approach, Tiba [9].

Here, we discuss another approach suggested by recent controllability–type results for elliptic systems [9], [1], [11]. It may be mainly compared with the mapping method since it reduces the optimal shape design problem to a control in the coefficients problem. However, no global description of the boundary of the variable domains is needed and no scaling has to be performed.

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Our basic idea is a simple one: if Ω is a subset of a fixed domain D, then it is possible to find some mapping $p : D \to \mathbb{R}$ (by an exact controllability–type argument [10]) such that $p > 0$ in Ω, $p = 0$ on $\partial\Omega$ and $p < 0$ in $D \setminus \overline{\Omega}$. Then the Heaviside mapping $H : \mathbb{R} \to \mathbb{R}$

$$(1) \qquad H(p) = \begin{cases} 1, & p \geq 0, \\ 0, & p < 0, \end{cases}$$

is the characteristic function of $\overline{\Omega}$ in D. We obtain a good approximation of it by the use of the Yosida approximation H_ε of the maximal monotone extension of H in $\mathbb{R} \times \mathbb{R}$:

$$(2) \qquad H_\varepsilon(p) = \begin{cases} 1, & p > \varepsilon, \\ \frac{p}{\varepsilon}, & p \in [0, \varepsilon], \\ 0, & p < 0\ . \end{cases}$$

In the sequel, we shall apply this approach to a model problem discussed by Céa [3] and Pironneau [8, Ch. 8]. In section 2, we perform a brief theoretical analysis of the proposed method and some numerical results are given in section 3.

Finally, we mention that our investigation may be also compared with the topology optimization method as described in the recent work of Bendsøe and Rodrigues [2].

2. The main results

We study the following optimization problem, denoted (P):

$$(3) \qquad \underset{\Omega \subset D}{\text{minimize}} \int_E |y_\Omega - y_d|^2 \, dx \qquad (P)$$

subject to the transmission problem

$$(4) \qquad -a_1 \Delta y_1 + a_0 y_1 = f \text{ in } \Omega,$$

$$(5) \qquad -a_2 \Delta y_2 + a_0 y_2 = f \text{ in } D \setminus \overline{\Omega}$$

$$(6) \qquad a_1 \frac{\partial y_1}{\partial n} = a_2 \frac{\partial y_2}{\partial n}; \quad y_1 = y_2 \text{ in } \partial\Omega \setminus (\partial\Omega \cap \partial D),$$

$$(7) \qquad a_i \frac{\partial y_i}{\partial n} = 0 \quad \text{in } \partial D,\ i = 1, 2\ .$$

Above a_0, a_1, a_2 are positive constants, $\frac{\partial}{\partial n}$ denotes the exterior normal derivative to Ω or D, $y_d \in L^2(E)$, $f \in L^2(\Omega)$, $E \subset D$ is a fixed measurable subset and $y_\Omega \in H^1(\Omega)$ is given by

$$(8) \qquad y_\Omega(x) = \begin{cases} y_1(x) & \text{in } \Omega, \\ y_2(x) & \text{in } D \setminus \Omega\ . \end{cases}$$

If χ is the characteristic function of Ω in D, then the variational formulation of the problem (4)–(7) is given by

$$(9) \qquad \int_D ([a_1 \chi + a_2(1 - \chi)] \nabla y_\Omega \nabla w + a_0 y_\Omega w - f w) \, dx \quad \forall\, w \in H^1(D)$$

and, in [8], it is analysed the case when

$$\chi \in \{g:\ D \to \mathbb{R};\ g(x) = 0 \text{ or } g(x) = 1 \quad \forall\, x \in D\} \tag{10}$$

is the control parameter. But the form of the constraint (10) makes the problem difficult to handle. Instead, we replace (9) by

$$\int_D ([a_1 H(p) + a_2(1 - H(p))]\nabla y_\Omega \nabla w + a_0 y_\Omega w - f w)\ dx = 0, \quad \forall\, w \in H^1(D) \tag{11}$$

and we obtain an unconstrained control problem (3), (11).

REMARK 2.1. Obviously, $H(p)$ is measurable for any measurable p, so $H(p) \in L^\infty(D)$. In [10], it is assumed that Ω is the image under a regular bijection $B:\ \mathbb{R}^n \to \mathbb{R}^n$ of the unit sphere and it is shown that one can find $p \in C^2(D)$ such that $p > 0$ in Ω, $p = 0$ in $\partial\Omega$ and $p < 0$ in $D \setminus \overline{\Omega}$. This is also valid for more general Ω (not connected, for instance). Since for any measurable p, the set $\Omega = \{x \in D \mid\ H(p) \geq 1\} = \{x \in D \mid\ p(x) \geq 0\}$ is measurable, we rennounce the regularity conditions on p, equivalently on the variable domain Ω.

These considerations show that we allow Ω to belong to a class including the images of the unit sphere in $\mathbb{R}^n$ under regular bijections, which is specific to the mapping method.

In this general setting, the interpretation of (9) as a transmission problem (4)–(7) is not always possible and we work directly with the variational formulation (11). The existence of a unique solution $y_\Omega \in H^1(D)$ is obvious, Lions and Magenes [6, Ch. II].

We approximate (11) by

$$\int_D ([a_1 H_\varepsilon(p) + a_2(1 - H_\varepsilon(p))]\nabla y \nabla w + a_0 y w - f w)\ dx = 0, \quad \forall w \in H^1(D)\ , \tag{12}$$

where H_ε is defined by (2).

THEOREM 2.2. *Let $y_\varepsilon \in H^1(D)$ be the unique solution of (12). Then*

$$y_\varepsilon \to y_\Omega \text{ strongly in } H^1(D), \tag{13}$$

when $\varepsilon \to 0$.

PROOF: We take $w = y_\varepsilon$ in (12). By the inequality

$$a_1 H_\varepsilon(p) + a_2(1 - H_\varepsilon(p)) \geq a = \min(a_0, a_1, a_2) > 0, \tag{14}$$

it yields that $\{y_\varepsilon\}$ is bounded in $H^1(D)$. We denote $y_\varepsilon \to \tilde{y}$ weakly in $H^1(D)$ and strongly in $L^2(D)$.

We notice that $a_1 H_\varepsilon(p) + a_2(1 - H_\varepsilon(p)) \to a_1 H(p) + a_2(1 - H(p))$ a.e. D and, since it is obviously bounded in $L^\infty(\Omega)$, the Lebesgue theorem gives $a_1 H_\varepsilon(p) + a_2(1 - H_\varepsilon(p)) \to a_1 H(p) + a_2(1 - H(p))$ strongly in $L^2(D)$, for instance. Then, clearly

$$[a_1 H_\varepsilon(p) + a_2(1 - H_\varepsilon(p))]\nabla y_\varepsilon \to [a_1 H(p) + a_2(1 - H(p))]\nabla \tilde{y}, \tag{15}$$

weakly in $L^2(D)$. Passing to the limit in (12), we get $\tilde{y} = y_\Omega$.

To establish (13), we start with the following inequality

$$0 \le a \int_D |\nabla(y_\varepsilon - y_\Omega)|^2 \, dx \le \int_D [a_1 H_\varepsilon(p) + a_2(1 - H_\varepsilon(p))] |\nabla(y_\varepsilon - y_\Omega)|^2 \, dx =$$

$$(16) \qquad \int_D [a_1 H_\varepsilon(p) + a_2(1 - H_\varepsilon(p))] \nabla y_\Omega \nabla(y_\Omega - y_\varepsilon) \, dx +$$

$$\int_D [a_1 H_\varepsilon(p) + a_2(1 - H_\varepsilon(p))] \nabla y_\varepsilon \nabla(y_\varepsilon - y_\Omega) \, dx = I_1^\varepsilon + I_2^\varepsilon \, .$$

By (15), we see that $I_1^\varepsilon \to 0$, while for I_2^ε we use (12) with $w = y_\varepsilon - y_\Omega$:

$$(17) \qquad I_2^\varepsilon = \int_\Omega [a_0 y_\varepsilon (y_\Omega - y_\varepsilon) + f(y_\varepsilon - y_\Omega)] \, dx \to 0 \, .$$

Combining (16), (17), we get (13) and finish the proof.

We approximate the problem (3), (9) by the following one:

$$(P_\varepsilon) \qquad \text{minimize} \int_E |y - y_d|^2 \, dx,$$

subject to any measurable p and $y \in H^1(D)$ given by (12).

REMARK 2.3. Generally, in the absence of some compactness assumption on the class of subdomains Ω (for instance ε-cone property, Pironneau [8, Ch.3], one may not obtain the existence of a solution for the problem (3)–(7). The same is valid for the problem (P_ε) since there are no coercivity conditions on the control parameter p. Obviously, one may ask a boundedness condition on p, $|p(x)| \le 1$, due to the relationship between p and Ω. But this does not imply existence since the weak limit in $L^\infty(D)$ of a sequence of characteristic functions is not necessarily a characteristic function.

We denote $[y^\varepsilon, p_\varepsilon]$ an δ-optimal pair of the problem (P_ε), $\delta > 0$, that is:

$$(18) \qquad J_\varepsilon(y^\varepsilon, p_\varepsilon) \le \inf(P_\varepsilon) + \delta,$$

where

$$J_\varepsilon(y^\varepsilon, p_\varepsilon) = \int_E |y^\varepsilon - y_d|^2 \, dx$$

and

$$(19) \quad \int_D ([a_1 H_\varepsilon(p_\varepsilon) + a_2(1 - H_\varepsilon(p_\varepsilon))] \nabla y^\varepsilon \nabla w + a_0 y^\varepsilon w - f w) \, dx = 0, \quad \forall w \in H^1(D) \, .$$

PROPOSITION 2.4. *For $\varepsilon \to 0$, we have $y^\varepsilon \to \tilde{y}$ strongly in $L^2(D)$ and weakly in $H^1(D)$ on a subsequence, such that*

$$(20) \qquad \int_E (\tilde{y} - y_d)^2 \, dx \le \inf(P) + \delta$$

PROOF: By definition, we get

$$J_\varepsilon(y^\varepsilon, p_\varepsilon) \leq \inf(P_\varepsilon) + \delta \leq J_\varepsilon(y_\varepsilon, p) + \delta, \tag{21}$$

for any admissible p and for y_ε given by (12). The same estimates as in the proof of Theorem 2.2, applied to (19), yield that $\{y^\varepsilon\}$ is bounded in $H^1(D)$. On a subsequence, we may assume that $y^\varepsilon \to \tilde{y}$ strongly in $L^2(D)$ and weakly in $H^1(D)$. Therefore, the left hand–side in (21) converges to the left hand–side from (20). As concerns the right hand–side in (21), we use Theorem 2.2 again and next we take the infimum on p, which ends the proof.

REMARK 2.5. By the properties of y^ε, we also obtain that

$$J_\varepsilon(y^\varepsilon, p_\varepsilon) \leq \inf(P) + 2\delta$$

for ε sufficiently small. In order to strengthen the above proposition, some relationship between y^ε and the solution of (11) associated to p_ε, stronger than Theorem 2.2, would be useful.

Let us assume now that H_ε is a C^1 approximation of H. Generally, this may be obtained by applying a Friedrichs mollifier in formula (2). A specific construction will be indicated in the next section.

In order to solve the problem (P_ε) by a gradient–type method, we discuss the adjoint system. Let $\theta_\varepsilon : L^\infty(D) \to L^2(D)$ be the approximate state mapping $p \to y$ defined by (12) (we restrict p to be in $L^\infty(D)$ here).

PROPOSITION 2.6. *θ_ε is a Gateaux differentiable mapping and $\nabla\theta_\varepsilon(p)v = r$ satisfies*

$$\int_D [(a_1 - a_2)H_\varepsilon'(p)v\nabla y \cdot \nabla w + (a_1 H_\varepsilon(p) + a_2(1 - H_\varepsilon(p)))\nabla r \cdot \nabla w + a_0 r w]\, dx = 0, \quad \forall w \in H^1(D), \tag{22}$$

where $y = \theta_\varepsilon(p) \in H^1(D)$ and v is arbitrary fixed in $L^\infty(D)$. Moreover $r \in H^1(D)$.

PROOF: Denote $y_\lambda = \theta_\varepsilon(p + \lambda v)$, $\lambda > 0$ and subtract the two equations corresponding to y, y_λ:

$$\begin{aligned} &\int_D \Big([a_1 H_\varepsilon(p+\lambda v) - a_1 H_\varepsilon(p) + a_2 H_\varepsilon(p) - a_2 H_\varepsilon(p+\lambda v)]\nabla y \nabla w + \\ &[a_1 H_\varepsilon(p+\lambda v) + a_2(1 - H_\varepsilon(p+\lambda v))]\nabla(y_\lambda - y)\nabla w + a_0(y_\lambda - y)w \Big)\, dx = 0, \\ &\text{for any } w \in H^1(D)\,. \end{aligned} \tag{23}$$

We divide by $\lambda > 0$ in (23) and give to w the value $\frac{y_\lambda - y}{\lambda}$. We obtain the following inequality:

$$a\left|\frac{y_\lambda - y}{\lambda}\right|^2_{H^1(D)} \leq |a_1 - a_2|\frac{1}{\varepsilon}|v|_{L^\infty(D)}|\nabla y|_{L^2(D)} \cdot \left|\nabla\frac{y_\lambda - y}{\lambda}\right|_{L^2(D)}, \tag{24}$$

where we also use the property that H_ε is Lipschitzian of constant $\frac{1}{\varepsilon}$. then (24) yields that $\{\frac{y_\lambda - y}{\lambda}\}$ is bounded in $H^1(D)$ with respect to $\lambda > 0$.

Let r be the limit in $L^2(D)$ strong or $H^1(D)$ weak of $\frac{y_\lambda - y}{\lambda}$. Passing to the limit in (23) (divided by λ), we obtain (22) and finish the proof.

We define the operator $T: L^2(D) \to L^1(D)$ linear, bounded, by $Tq = \ell$, where

$$\ell = (a_1 - a_2)H_\varepsilon'(p)\nabla z \cdot \nabla y \tag{25}$$

and $z \in H^1(D)$ is the solution of the adjoint equation

$$\int_D ([a_1 H_\varepsilon(p) + a_2(1 - H_\varepsilon(p))]\nabla z \cdot \nabla w + a_0 z w + q w)\, dx = 0, \quad \forall w \in H^1(D)\,. \tag{26}$$

It turns out that the linear, continuous operator $S: L^\infty(D) \to L^2(D)$, by $Sv = \nabla\theta_\varepsilon(p)v = r$, is just the adjoint of the operator T. This may be inferred by the definition, by choosing $w = r$ in (26) and $w = z$ in (22).

This enables us to compute the gradient of the cost functional of (P_ε), which will be useful in the next section (we denote shortly $J_\varepsilon(p)$ instead of $J_\varepsilon(y,p)$, $y = \theta_\varepsilon(p)$):

$$\begin{aligned} \lim_{\lambda\to 0} \frac{J_\varepsilon(p+\lambda v) - J_\varepsilon(p)}{\lambda} &= 2\int_E r(y - y_d)\, dx = \\ &2\int_D r\chi_E(y - y_d)\, dx = 2(Sv, \chi_E(y-y_d))_{L^2(D)} = \\ &2(v, T\chi_E(y - y_d))_{L^\infty(D)\times L(D)}\,, \quad \forall v \in L^\infty(D)\,. \end{aligned} \tag{27}$$

Here χ_E is the characteristic function of E in D and to finish the calculation one has to put $q = \chi_E(y - y_d)$ in (25), (26).

3. Numerical examples

In this section we choose $D = E = (0,1)\times(0,1)$, $a_1 = 10$, $a_2 = a_0 = 1$ and $f = x_1^2 + x_2^2$. The Heaviside mapping is approximated by

$$H_\varepsilon(p) = \begin{cases} 1 - \frac{1}{2}e^{-\frac{p}{\varepsilon}}, & p \ge 0\,, \\ \frac{1}{2}e^{\frac{p}{\varepsilon}}, & p < 0\,. \end{cases} \tag{28}$$

The regularized state problem (12) is discretized by finite element method. In discretization four node quadrilateral elements are used. The control parameter p in (P_ε) is taken to be piecewise constant. The discrete analogue of (12) is then the linear system of equations

$$K(p)q = f, \tag{29}$$

where q is the vector of nodal values of the state solution, $K(p)$ is the "stiffness" matrix (depending on the control variable p) and f is the "force" vector.

The discrete analogue of (P_ε) then reads

$$(P_\varepsilon^h) \qquad \text{minimize } (q - q_d)^T M (q - q_d)$$

subject to (29). Here q_d is the vector containing the nodal values of y_d and M is the "mass" matrix. In the numerical solution of (P_ε^h) we have used the pre-conditioned, limited memory quasi–Newton conjugate gradient algorithm from the NAG–subroutine Library. The gradient of the cost is obtained using the discrete analogues of (26) and (27).

Problem (P_ε^h) is highly nonlinear and nonconvex. Therefore a good initial guess is needed to be able to find the global minimum. If one do not know the topology of the optimal domain the choice for initial guess may be difficult. However, the choice $p = 0$ has proved to be extremely efficient. From (28) it follows that $H_\varepsilon(0) = \frac{1}{2}$ so the state problem reduces to

$$(30) \qquad \begin{cases} -\tilde{a}\Delta y + a_0 y = f \text{ in } D \\ \tilde{a}\dfrac{\partial y}{\partial n} = 0 \text{ on } \partial D\,, \end{cases}$$

where $\tilde{a} = (a_1 + a_2)/2$. This corresponds to a "homogenized" design.

REMARK 3.1. Choosing $0 < a_2 << a_1$ corresponds to the material distribution problem in structural optimization. Namely those regions $\Omega \backslash \overline{D}$ with very low thickness a_2 may be removed from the structure without weakening it.

REMARK 3.2 Another approach to get rid of the constraint (10) is to introduce a control problem of the form

$$(P') \qquad \text{minimize} \int_E |y - y_d|^2 \, dx$$

subject to $a_2 \leq p \leq a_1$, $p \in L^\infty(D)$ and

$$(31) \qquad \begin{cases} -\nabla \cdot (p \nabla y) + a_0 y = f \text{ in } D \\ p\dfrac{\partial y}{\partial n} = 0 \text{ on } \partial D. \end{cases}$$

This approach is very similar to that presented in [**2**]. One may then set $\Omega = \{x \in D \mid p(x) = a_1\}$. However, the set $\{x \in D \mid a_2 < p < a_1\}$ is generally non-empty, which may make it difficult to interpret the results. The discrete analogue (P_h') of (P') can be formulated in the similar way as of (P_ε). However, (P_h') is a constrained optimization problem.

Finally, we present some numerical results that were computed on HP9000/345–computer with double precision arithmetic. In computations the cost function was scaled with a factor 0.5×10^5 and the regularization parameter $\varepsilon = 1/10$ was used. As the discrete control parameter p is piecewise constant, nodal averaging of it was done for plotting purposes.

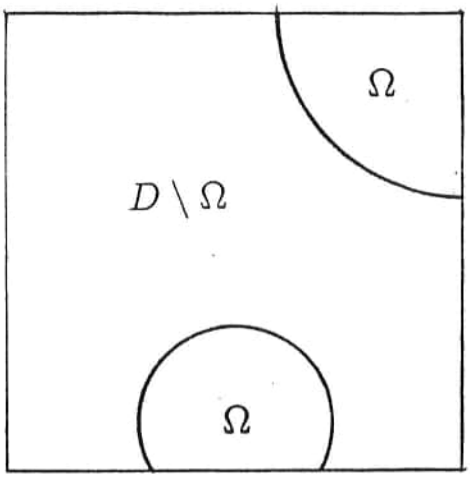

Figure 3.1

Example 3.1. Let y_d be the solution of the transmission problem in the geometry shown in Figure 3.1. Two runs with 100 and 900 finite elements were done. In both cases initial guess $p = 0$ was used. The results are shown in Table 3.1 and Figures 3.2–3.3 (meshes and contour $p = 0$). In both cases the global optimums were clearly found.

Number of elements	Initial cost	Final cost	Iterations	CPU-seconds
100	23.1	1.41×10^{-3}	15	29
900	24.9	1.11×10^{-3}	23	395

Table 3.1

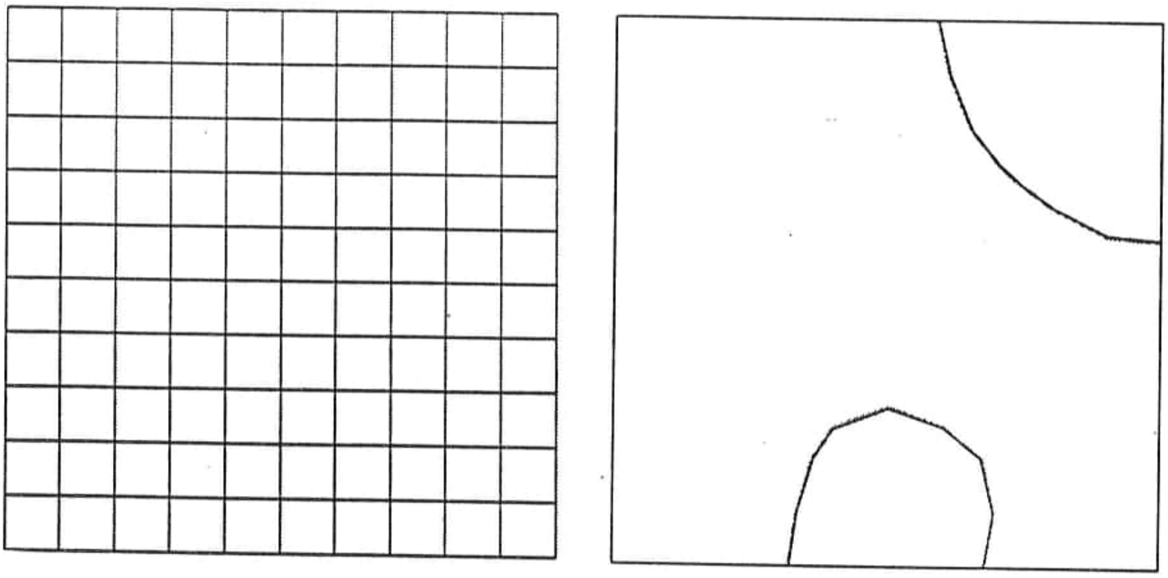

Figure 3.2

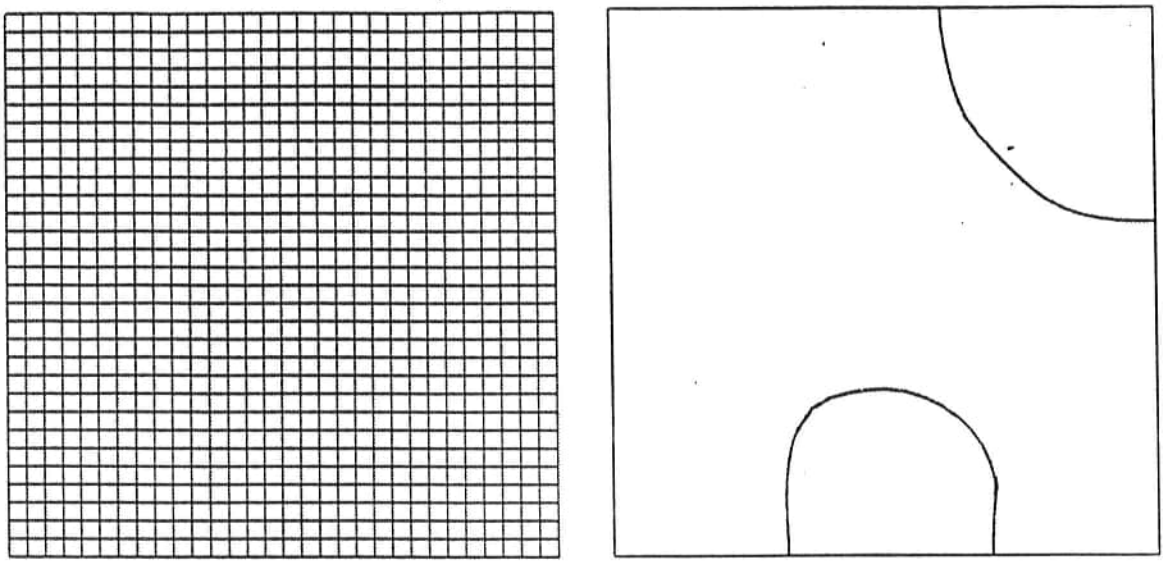

Figure 3.3

Example 3.2. Let y_d be the solution of the transmission problem in the geometry shown in Figure 3.4. Again two runs with the same meshes as in the previous example were done. The results are shown in Table 3.2 and Figures 3.5–3.6. The situation here is more complicated than in the previous example. However, the results are acceptable.

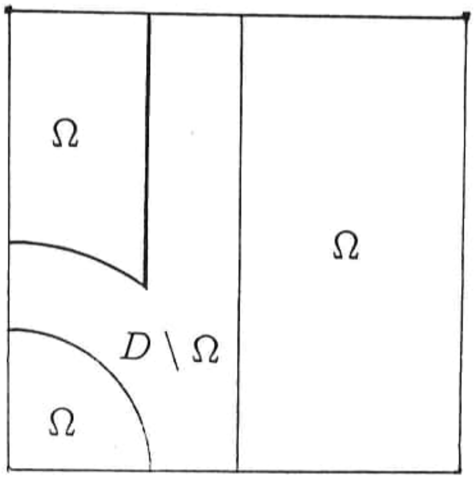

Figure 3.4

Number of elements	Initial cost	Final cost	Iterations	CPU-seconds
100	2.84	7.69×10^{-4}	20	34
900	3.07	8.07×10^{-3}	41	680

Table 3.2

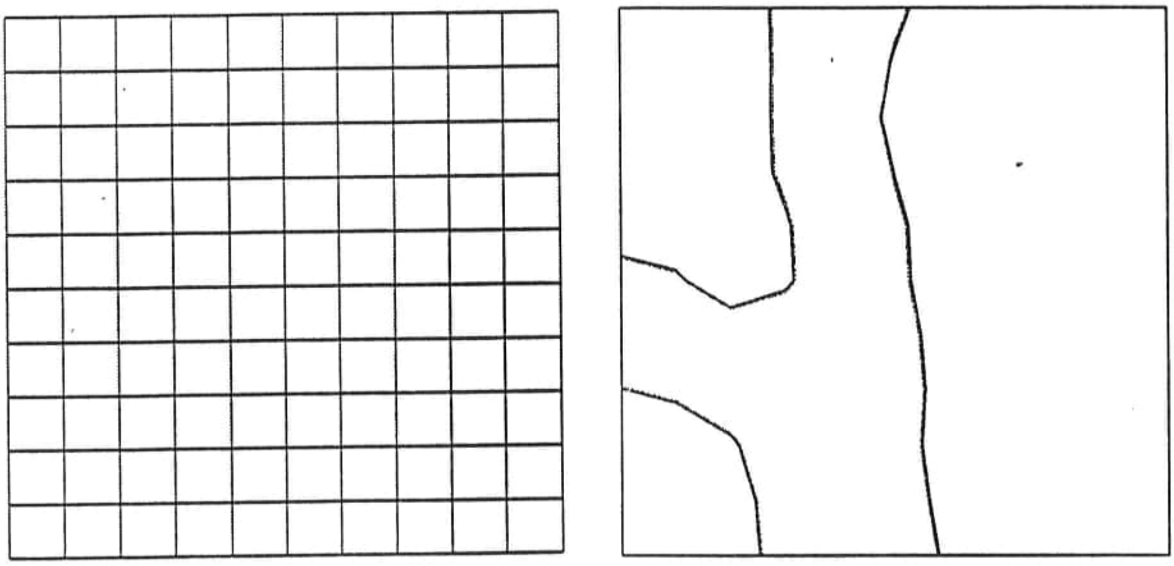

Figure 3.5.

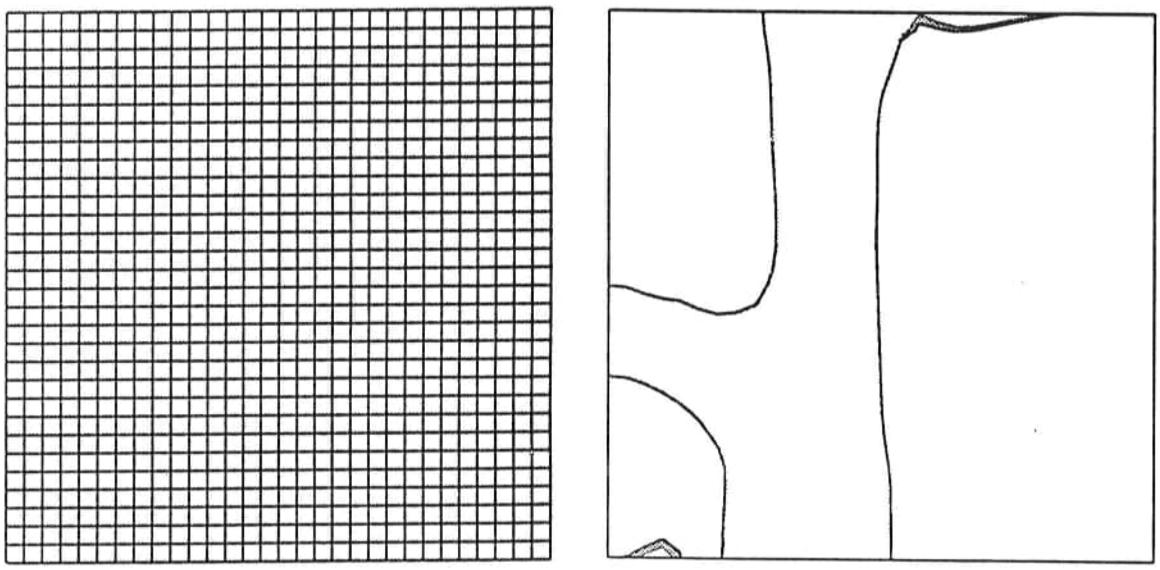

Figure 3.6.

Example 3.3. Let y_d be as in Example 3.1. Let $\Omega_2 =]0,1[\times]0,\frac{1}{4}[\cup]0,1[\times]\frac{3}{4},1[$ and $\Omega_B = B(x_0,\frac{1}{4})$, $x_0 = (\frac{1}{2},\frac{1}{2})$. Two runs with 400 finite elements were done with different initial guesses

$$p = \begin{cases} 0.2 \text{ in } \Omega^0 \\ -0.2 \text{ in } D \setminus \Omega^0 \end{cases}, \qquad \Omega^0 = \Omega_2,\ \Omega_B.$$

The results are shown in Table 3.3 and Figures 3.7–3.8. In both cases only a local minimum was achieved. However, the topology was changed. The choice of $|p| = 0.2$ does not imply that $H_\varepsilon(p)$ is a very good approximation of the Heaviside function. However, choosing $|p|$ "too large" does not yield convergence to any sensible solution.

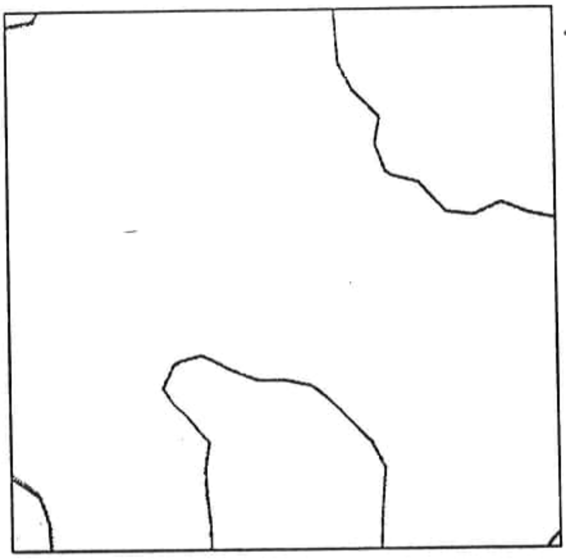

Figure 3.7

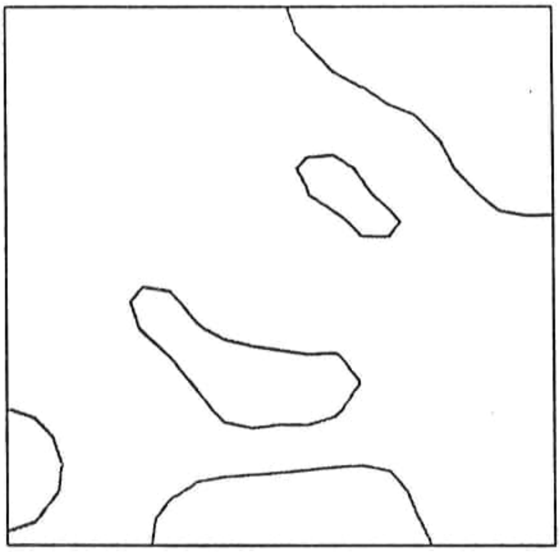

Figure 3.8

Ω^0	Initial cost	Final cost	Iterations	CPU-seconds
Ω_2	11.2	1.94×10^{-2}	43	311
Ω_B	13.3	2.12×10^{-1}	50	410

Table 3.3

Example 3.4. In order to make some comparisons, two test runs with problem (P_h') were performed. In both runs a finite element mesh with 100 elements was used. As (P_h') is a constrained optimization problem we used a SQP-algorithm from the NAG-library in solving it. The results are shown in Table 3.4 and Figures 3.9–3.10. In both cases the topology of the optimal domain can be inferred from the results. However, one could expect more "sharpness" from the control parameter p. The convergence of the iteration was also slower than in the case of (P_ε^h).

Initial guess	Initial cost	Final cost	Iterations	CPU-seconds
$p = 5.5$	23.1	9.07×10^{-3}	110	420
$p = a_1$ in Ω_B $p = a_2$ in $D \setminus \Omega_B$	5.94	3.44×10^{-3}	123	468

Table 3.4

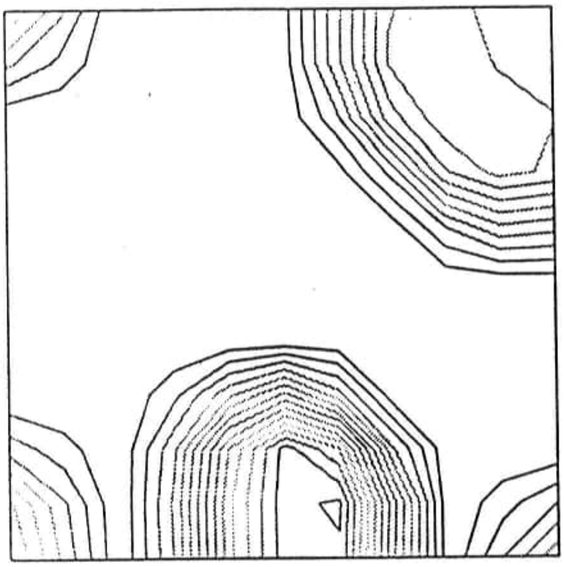

Figure 3.9

4. Conclusions

A fixed domain approach for solving a shape optimization problem was introduced and analyzed. The proposed method is very simple to implement on computer compared with the conventional boundary variation method. No complicated moving finite element meshes and a priori knowledge of the topology of the optimal domain are needed. Although gradient based method was used in optimization, the gradient computations are very easy compared with those of boundary variation methods. This makes the method very cost-effective in terms of man-hours and computing time.

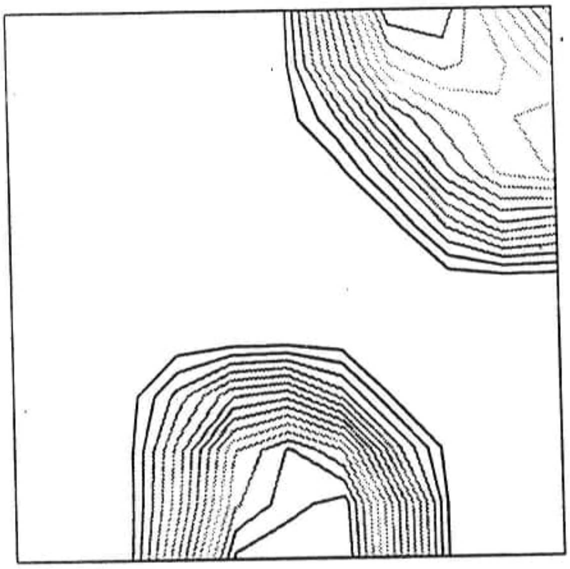

Figure 3.10

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