Modeling the Effects of Acoustic Visco–Thermal Boundary Layers as a Wentzell Boundary Condition

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August 8, 2018

Acoustic wave propagation in air

Most common model: the **linear wave equation** for the acoustic pressure:

$$\frac{\partial^2 P}{\partial t^2} - c_0^2 \Delta P = 0$$

In frequency domain, the **Helmholtz equation**:

$$-k_0^2 p - \Delta p = 0,$$

where $P(t, \boldsymbol{x}) = \operatorname{Re} e^{i\omega t} p(\boldsymbol{x})$ and $k_0 = \dfrac{\omega}{c_0}$

Let us review the basic assumptions!

Modeling assumptions in “standard” acoustics

1. **Linearity**: acoustic pressure (p), velocity ($\boldsymbol{u}$), temperature (T) small disturbances around constant base states p_0, $\boldsymbol{u}_0$, T_0 (**Three** acoustic fields in general!)
2. **Still air** ($\boldsymbol{u}_0 = \boldsymbol{0}$)
3. Sound propagation is **isentropic** (adiabatic and reversible)

   Isentropy $\Rightarrow$

   - Acoustic velocity: $\boldsymbol{u} = \dfrac{\mathrm{i}}{\omega \rho_0} \nabla p$
   - Temperature fluctuations $T = \dfrac{T_0}{\rho_0} \dfrac{(\gamma - 1)}{\gamma} p$
   - Density fluctuations $\rho = \dfrac{1}{c_0^2} p$

   Thus, isentropy $\Rightarrow$ only **one** scalar field needs to be computed!

How realistic is the isentropic assumption?

- No losses (viscous, thermal) taken into account, by definition
- Loss mechanisms in acoustics:
  - "Bulk losses" can usually be neglected in comparison with
  - **interaction with solid surfaces**
- Thermal interaction:
  - Thermal conduction in a solid $\gg$ thermal conduction in air. **Isothermal wall** is the standard assumption ($\Rightarrow T = 0$ at wall)
  - Diffusion of acoustic thermal oscillations close to wall (thermal conductivity coefficient κ)
- Viscous interaction:
  - Non-slip condition $\boldsymbol{u} = \mathbf{0}$ at wall
  - Diffusion of tangential momentum close to wall (viscosity coefficient ν)
- Thus, interaction with solid surface creates thermal and viscous **boundary layers**

Modeling visco–thermal losses

When are visco–thermal losses important?

- Sound propagation in **long narrow wave guides**: musical instruments, measurement devices
- Sound propagation in **small devices**: microphones, hearing aids, micro-speakers
- Modeling air losses for Micro Electro Mechanical (MEMS) sensors

$$\text{In summary: when } \frac{\text{Solid Surface Area}}{\text{Total Air Volume}} \text{ not too small}$$

The linearized, compressible Navier–Stokes equations

Includes visco–thermal losses

$$\mathrm{i}\omega\rho + \rho_0 \nabla \cdot \boldsymbol{U} = 0, \qquad \text{(Mass Conservation)}$$

$$\mathrm{i}\omega \boldsymbol{U} + \frac{1}{\rho_0}\nabla p - \nu\left(\Delta \boldsymbol{U} + \frac{1}{3}\nabla(\nabla \cdot \boldsymbol{U})\right) = \boldsymbol{0}, \qquad \text{(Momentum Balance)}$$

$$\mathrm{i}\omega\rho_0 c_V T + p_0 \nabla \cdot \boldsymbol{U} - \kappa \Delta T = 0, \qquad \text{(Energy Balance)}$$

ν, κ: viscosity coefficient, thermal conductivity of air

c_V: heat capacity, constant volume

Linearized equation of state (ideal gas law)

$$\frac{\rho}{\rho_0} = \frac{p}{p_0} + \frac{T}{T_0}$$

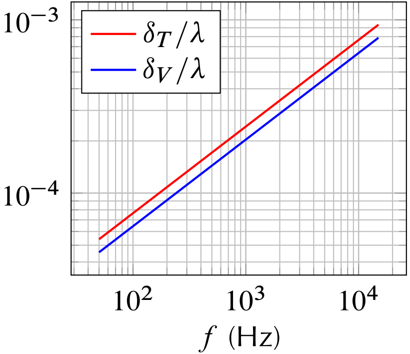
Boundary conditions at solid walls:

$$\boldsymbol{U} = \boldsymbol{0} \qquad T = 0$$

Why not simply use this model?

Acoustic boundary layer thickness

Thickness of thermal (δ_T) and viscous (δ_V) layers in terms of wave length (λ) for audio frequencies



- Helmholtz equation: $\sim$ 10 grid points per wave length
- Thus, a Helmholtz grid cannot resolve the boundary layer
- Direct modeling with compressible Navier–Stokes very computationally demanding
- Recommended (e.g. by Comsol, Acoustics Module) to be used only in hybrid modeling

Modeling visco–thermal losses

Basically two approaches in literature:

1. Waveguide techniques
   - Many contributions, starting already with Kirchhoff (1868!)
   - Linearized Navier–Stokes equations
   - Exact or approximate **modal** solutions in wave guides (one axial space coordinate z)
   - Yields a complex wave number k in transversal average pressure $p(z) = p(0)\,\mathrm{e}^{\mathrm{i}kz}$ (dispersion relation $k = k(\omega)$)
2. Boundary-layer theory
   - A version of Prantl's boundary-layer technique for oscillatory exterior flow
   - Applied to linearized, compressible Navier–Stokes equations
   - Suggested as a **post-processing** approach to estimate total losses from isentropic pressure data (e.g. Searby et al., *J. Propul. Power* (2008))
   - Iterative procedure: R. Bossart, N. Joly, M. Bruneau, *J. Sound Vibration* (2003)

Our approach – outline

- Boundary-layer analysis ⇒ explicit formulas for velocity and density fluctuations in boundary layer
- Fluctuations exponentially attain isentropic conditions outside of boundary layer
- Rewrite exact mass conservation law in boundary layer to the same form as in the isentropic case **but with a modified wall boundary condition**
- Suggests the use of Helmholtz equation with the modified boundary condition for visco–thermal analysis
    - Modified problem well posed; easy FE implementation
    - Equivalent to classical expressions for special geometries
    - Generally applicable to most acoustic problems
    - Test case: results match closely Navier–Stokes solutions to a much lower computational cost

Isentropic approximations

Linearized, compressible Navier–Stokes equations:

$$\mathrm{i}\omega\rho + \rho_0 \nabla\cdot\boldsymbol{U} = 0, \qquad \text{(mass)}$$

$$\mathrm{i}\omega\boldsymbol{U} + \frac{1}{\rho_0}\nabla p - \nu\big(\Delta\boldsymbol{U} + \frac{1}{3}\nabla(\nabla\cdot\boldsymbol{U})\big) = \boldsymbol{0}, \qquad \text{(mom)}$$

$$\mathrm{i}\omega\rho_0 c_V T + p_0\nabla\cdot\boldsymbol{U} - \kappa\Delta T = 0, \qquad \text{(energy)}$$

Equation of state: $\rho/\rho_0 = p/p_0 + T/T_0$.

Boundary conditions at solid walls: $\boldsymbol{U} = \boldsymbol{0}$, $T = 0$.

Isentropic assumptions $\Rightarrow$ $\nu = \kappa = 0$ (no boundary layer) , $p = c_0^2\rho$, and

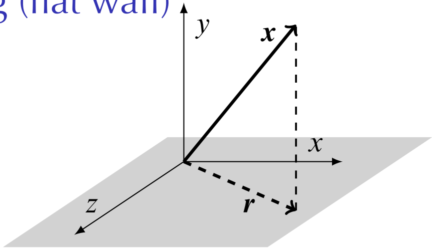
$$\frac{\mathrm{i}\omega}{c_0^2} p + \rho_0\nabla\cdot\boldsymbol{U} = 0, \qquad \text{(mass)}$$

$$\mathrm{i}\omega\boldsymbol{U} + \frac{1}{\rho_0}\nabla p = \boldsymbol{0}, \qquad \text{(mom)}$$

$$\mathrm{i}\omega\rho_0 c_V T + p_0\nabla\cdot\boldsymbol{U} = 0 \qquad \text{(energy)}$$

Boundary conditions at solid walls: $\boldsymbol{n}\cdot\boldsymbol{U} = 0$

Normal and tangential splitting (flat wall)



- Coordinates $\boldsymbol{x} = (x, y, z)$; velocity $\boldsymbol{U} = (u, v, w)$
- Normal direction y, wall at $y = 0$
- Projections on wall plane: $\boldsymbol{r} = (x, 0, z)$, $\boldsymbol{u} = (u, 0, w)$
- Tangential gradient, divergence, and Laplace operators

$$\nabla_\mathsf{T} = \left(\frac{\partial}{\partial x}, 0, \frac{\partial}{\partial z}\right), \qquad \Delta_\mathsf{T} = \nabla_\mathsf{T} \cdot \nabla_\mathsf{T} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}.$$

- Then, for instance:

$$\nabla \cdot \boldsymbol{U} = \nabla_\mathsf{T} \cdot \boldsymbol{u} + \frac{\partial v}{\partial y}, \qquad \nabla T = \nabla_\mathsf{T} T + \left(0, \frac{\partial T}{\partial y}, 0\right)$$

$$\Delta T = \Delta_\mathsf{T} T + \frac{\partial^2 T}{\partial y^2}$$

Splitting, boundary-layer limits (flat wall)

Navier–Stokes Acoustic boundary layer equations

$$
\begin{aligned}
\mathrm{i}\omega\frac{\rho}{\rho_0} + \nabla_\mathsf{T}\cdot\boldsymbol{u} + \frac{\partial v}{\partial y} &= 0,\\
\mathrm{i}\omega\boldsymbol{u} + \frac{1}{\rho_0}\nabla_\mathsf{T} p - \nu\left(\Delta_\mathsf{T}\boldsymbol{u} + \frac{\partial^2\boldsymbol{u}}{\partial y^2} + \frac{1}{3}\nabla_\mathsf{T}\left(\nabla_\mathsf{T}\cdot\boldsymbol{u} + \frac{\partial v}{\partial y}\right)\right) &= \mathbf{0},\\
\mathrm{i}\omega v + \frac{1}{\rho_0}\frac{\partial p}{\partial y} - \nu\left(\Delta_\mathsf{T} v + \frac{\partial^2 v}{\partial y^2} + \frac{1}{3}\frac{\partial}{\partial y}\left(\nabla_\mathsf{T}\cdot\boldsymbol{u} + \frac{\partial v}{\partial y}\right)\right) &= 0\\
\mathrm{i}\omega\rho_0 c_V T + p_0\left(\nabla_\mathsf{T}\cdot\boldsymbol{u} + \frac{\partial v}{\partial y}\right) - \kappa\left(\Delta_\mathsf{T} T + \frac{\partial^2 T}{\partial y^2}\right) &= 0,
\end{aligned}
$$

- Length scales: $1/k_0$ (horizontal), δ (vertical)
- Rescaling variables, consider small values of ν, κ, δ, keeping only leading terms

Splitting, boundary-layer limits (flat wall)

Navier–Stokes Acoustic boundary layer equations

$$\mathrm{i}\omega\frac{\rho}{\rho_0} + \nabla_\mathsf{T}\cdot\boldsymbol{u} + \frac{\partial v}{\partial y} = 0,$$

$$\mathrm{i}\omega\boldsymbol{u} + \frac{1}{\rho_0}\nabla_\mathsf{T}p - \nu\left(\Delta_\mathsf{T}\boldsymbol{u} + \frac{\partial^2\boldsymbol{u}}{\partial y^2} + \frac{1}{3}\nabla_\mathsf{T}\left(\nabla_\mathsf{T}\cdot\boldsymbol{u} + \frac{\partial v}{\partial y}\right)\right) = \boldsymbol{0},$$

$$\mathrm{i}\omega v + \frac{1}{\rho_0}\frac{\partial p}{\partial y} - \nu\left(\Delta_\mathsf{T}v + \frac{\partial^2 v}{\partial y^2} + \frac{1}{3}\frac{\partial}{\partial y}\left(\nabla_\mathsf{T}\cdot\boldsymbol{u} + \frac{\partial v}{\partial y}\right)\right) = 0$$

$$\mathrm{i}\omega\rho_0 c_V T + p_0\left(\nabla_\mathsf{T}\cdot\boldsymbol{u} + \frac{\partial v}{\partial y}\right) - \kappa\left(\Delta_\mathsf{T}T + \frac{\partial^2 T}{\partial y^2}\right) = 0,$$

- Length scales: $1/k_0$ (horizontal), δ (vertical)
- Rescaling variables, consider small values of ν, κ, δ, keeping only leading terms

The acoustic boundary-layer equations

$$\mathrm{i}\omega\frac{\rho}{\rho_0} + \nabla_\mathsf{T}\cdot\boldsymbol{u} + \frac{\partial v}{\partial y} = 0,$$

$$\mathrm{i}\omega\boldsymbol{u} + \frac{1}{\rho_0}\nabla_\mathsf{T} p - \nu\frac{\partial^2\boldsymbol{u}}{\partial y^2} = \boldsymbol{0},$$

$$\frac{\partial p}{\partial y} = 0,$$

$$\mathrm{i}\omega\rho_0 c_V T + p_0\left(\nabla_\mathsf{T}\cdot\boldsymbol{u} + \frac{\partial v}{\partial y}\right) - \kappa\,\frac{\partial^2 T}{\partial y^2} = 0.$$

Approximations to the linearized Navier–Stokes equations for

- ν, κ small,
- for $y \sim O\left(\sqrt{\nu/\omega}, \sqrt{\kappa/\omega}\right)$

Boundary conditions at $y = 0$: $\boldsymbol{u} = 0$, $v = 0$, $T = 0$;
equation of state: $\rho/\rho_0 = p/p_0 + T/T_0$

As $y \to +\infty$, solutions approach isentropic fields satisfying

$$\boldsymbol{u}^\infty = \frac{\mathrm{i}}{\omega\rho_0}\nabla_\mathsf{T} p^\infty$$

$$\rho^\infty = \frac{1}{c_0^2}p^\infty, \qquad T^\infty = \frac{T_0}{\rho_0}\frac{(\gamma-1)}{\gamma}p^\infty$$

1: the viscous boundary layer

Tangential velocity $\boldsymbol{u}$ satisfies boundary-value problem

$$\begin{aligned} \mathrm{i}\omega \boldsymbol{u} - \nu \frac{\partial^2 \boldsymbol{u}}{\partial y^2} - \mathrm{i}\omega \boldsymbol{u}^\infty &= \boldsymbol{0}, \\ \boldsymbol{u}|_{y=0} &= \boldsymbol{0}, \\ \lim_{y\to+\infty} &= \boldsymbol{u}^\infty. \end{aligned}$$

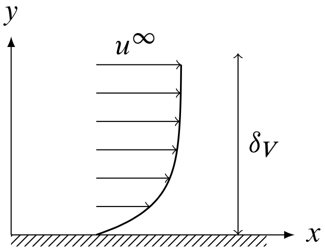
Solution (Stokes second problem):

$$\boldsymbol{u} = \boldsymbol{u}^\infty(\boldsymbol{r}) \left(1 - \mathrm{e}^{-(1+\mathrm{i})y/\delta_V}\right),$$

where

$$\delta_V = \sqrt{\frac{2\nu}{\omega}},$$

is the viscous boundary-layer thickness.



2: the thermal boundary layer

- Thermal boundary layer equations less straightforward to derive
- Our derivation a generalization of the 1D analysis of Rienstra & Hirschberg, *An Introduction to Acoustics* (2015)
- Equations rewritten by introducing the **excess density** $\rho_e = \rho - \rho^\infty$

2: the thermal boundary layer

$$
\begin{aligned}
\mathrm{i}\omega\rho_0 c_p \frac{\rho_e}{\rho_0} - \kappa \frac{\partial^2}{\partial y^2}\frac{\rho_e}{\rho_0} &= 0 && \text{for } y > 0,\\
\frac{\rho_e}{\rho_0} &\to 0 && \text{as } y \to +\infty,\\
\frac{\rho_e}{\rho_0} &= \frac{\gamma - 1}{\gamma}\frac{p^\infty}{p_0} && \text{at } y = 0.
\end{aligned}
$$

Solution:

$$
\frac{\rho_e}{\rho_0} = \frac{\gamma - 1}{\gamma}\frac{p^\infty(\boldsymbol{r})}{p_0}\mathrm{e}^{-(1+\mathrm{i})y/\delta_T},
$$

where

$$
\delta_T = \sqrt{\frac{2\kappa}{\omega\rho_0 c_p}}.
$$

is the thermal boundary-layer thickness.

From boundary layers to boundary conditions

Mass conservation law (boundary layer approximation or isentropic):

$$\mathrm{i}\omega \frac{\rho}{\rho_0} + \nabla_\mathrm{T} \cdot \boldsymbol{u} + \frac{\partial v}{\partial y} = 0$$

Integration in wall-normal direction $\Rightarrow$

$$\mathrm{i}\omega \int_0^{\tilde{y}} \frac{\rho}{\rho_0}\, dy + \int_0^{\tilde{y}} \nabla_\mathrm{T} \cdot \boldsymbol{u}\, dy + v|_{y=\tilde{y}} - v|_{y=0} = 0 \qquad \forall \tilde{y} \text{ s.t. } 0 < \tilde{y} \ll 1/k_0,$$

where $v|_{y=0} = 0$ from boundary conditions.

- The mass conservation law is the same in both cases!
- Boundary-layer case: strong gradients in ρ, $\boldsymbol{u}$
- Isentropic case ρ, $\boldsymbol{u}$ almost constant close to wall

From boundary layers to boundary conditions

Integrated mass conservation law using boundary-layer approximations:

$$\mathrm{i}\omega \int_0^{\tilde{y}} \frac{\rho}{\rho_0}\,dy + \int_0^{\tilde{y}} \nabla_{\mathsf{T}} \cdot \boldsymbol{u}\,dy + v|_{y=\tilde{y}} - v|_{y=0}$$

$$= \mathrm{i}\omega \int_0^{\tilde{y}} \frac{\rho^\infty}{\rho_0}\,dy + \mathrm{i}\omega \int_0^{\tilde{y}} \left(\frac{\rho}{\rho_0} - \frac{\rho^\infty}{\rho_0}\right) dy + \int_0^{\tilde{y}} \nabla_{\mathsf{T}} \cdot \boldsymbol{u}\,dy + v|_{y=\tilde{y}} - v|_{y=0}$$

$$= \big[\text{insert exact formulas...}\big]$$

$$= \mathrm{i}\omega \int_0^{\tilde{y}} \frac{\rho^\infty}{\rho_0}\,dy + \int_0^{\tilde{y}} \nabla_{\mathsf{T}} \cdot \boldsymbol{u}^\infty\,dy + \tilde{v}|_{y=\tilde{y}} - v_{\mathsf{W}} = 0$$

where

$$v_{\mathsf{W}} = -\delta_V \frac{\mathrm{i}-1}{2} \nabla_{\mathsf{T}} \cdot \boldsymbol{u}^\infty - \delta_T \frac{\omega(\gamma-1)(1+\mathrm{i})}{2\gamma p_0} p^\infty,$$

$$\tilde{v}|_{y=\tilde{y}} = v|_{y=\tilde{y}} + f(\tilde{y}), \qquad \big|f(\tilde{y})\big| \le C(\mathrm{e}^{-\tilde{y}/\delta_V} + \mathrm{e}^{-\tilde{y}/\delta_T})$$

From boundary layers to boundary conditions

- Thus, mass conservation law using boundary-layer approximations:

$$\mathrm{i}\omega \int_0^{\tilde{y}} \frac{\rho^\infty}{\rho_0}\, dy + \int_0^{\tilde{y}} \nabla_\mathrm{T} \cdot \boldsymbol{u}^\infty \, dy + \tilde{v}|_{y=\tilde{y}} - v_\mathrm{W} = 0 \qquad \forall \tilde{y} \text{ s.t. } 0 < \tilde{y} \ll 1/k_0,$$

  where

$$v_\mathrm{W} = -\delta_V \frac{\mathrm{i}-1}{2} \nabla_\mathrm{T} \cdot \boldsymbol{u}^\infty - \delta_T \frac{\omega(\gamma-1)(1+\mathrm{i})}{2\gamma p_0} p^\infty .$$

- Equal to the isentropic mass conservation law with $O(\delta_V + \delta_T)$-perturbed wall-normal velocity
- Idea: use isentropic model, but replace $v = 0$ with $v = v_W$ as boundary conditions at $y = 0$

From boundary layers to boundary conditions

$$v_{\mathsf{W}} = -\delta_V \frac{\mathrm{i}-1}{2} \nabla_{\mathsf{T}} \cdot \boldsymbol{u}^{\infty} - \delta_T \frac{\omega(\gamma-1)(1+\mathrm{i})}{2\gamma p_0} p^{\infty}$$

- Recall: when isetropic, $\boldsymbol{U}$ can be computed from p:

$$\boldsymbol{u}^{\infty} = \frac{\mathrm{i}}{\rho_0 c_0} \nabla_{\mathsf{T}} p^{\infty}$$
$$v^{\infty} = \frac{\mathrm{i}}{\rho_0 c_0} \frac{\partial p^{\infty}}{\partial n}$$

- Thus, setting $v = v_{\mathsf{W}}$ corresponds to boundary condition

$$\frac{\partial p^{\infty}}{\partial n} - \delta_V \frac{\mathrm{i}-1}{2} \Delta_{\mathsf{T}} p^{\infty} + \delta_T k_0^2 \frac{(\mathrm{i}-1)(\gamma-1)}{2} p^{\infty} = 0$$

- Constitutes an $O(\delta_V + \delta_T)$ perturbation of the hard-wall condition

$$\frac{\partial p^{\infty}}{\partial n} = 0$$

- A so-called Wentzell boundary condition

How about curved walls?

- A **flat wall** assumed in the above derivations
- For smooth non-flat surfaces, split using $\boldsymbol{U} = \boldsymbol{u} + (\boldsymbol{U} \cdot \boldsymbol{n})\boldsymbol{n}$ and curvlinear operators

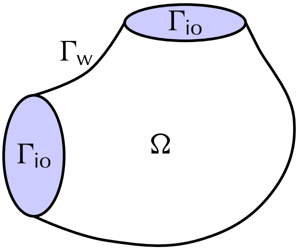
$$\nabla T = \nabla_{\mathsf{T}} T + \boldsymbol{n}\frac{\partial T}{\partial n}, \qquad \nabla \cdot \boldsymbol{U} = \nabla_{\mathsf{T}} \cdot \boldsymbol{u} + \frac{\partial(\boldsymbol{U} \cdot \boldsymbol{n})}{\partial n},$$
$$\Delta_{\mathsf{T}} T = \nabla_{\mathsf{T}} \cdot \nabla_{\mathsf{T}} T.$$

- Would in general involve wall-curvature effects, e.g.

$$\Delta T = \Delta_{\mathsf{T}} T + \frac{\partial^2 T}{\partial n^2} + \kappa \frac{\partial T}{\partial n},$$

  $\kappa = \nabla_{\mathsf{T}} \cdot \boldsymbol{n}$ is (twice) the mean curvature of the wall
- However, δ_T, $\delta_V \sim 20 - 400\ \mu$m for air in audio range
- Radii of curvature for smooth walls typically $\gg \delta_T, \delta_V$
- Thus, often reasonable to ignore curvature effects

Example problem: an acoustic cavity



$$
\begin{aligned}
-k_0^2 p - \Delta p &= 0 && \text{in } \Omega, \\
\mathrm{i}k_0 p + \frac{\partial p}{\partial n} &= 2\mathrm{i}k_0 g && \text{on } \Gamma_{io}, \\
-\delta_V \frac{\mathrm{i}-1}{2} \Delta_\mathrm{T} p + \delta_T k_0^2 \frac{(\mathrm{i}-1)(\gamma-1)}{2} p + \frac{\partial p}{\partial n} &= 0 && \text{on } \Gamma_w, \\
\boldsymbol{n}_\mathrm{T} \cdot \nabla_\mathrm{T} p &= 0 && \text{on } \partial\Gamma_w,
\end{aligned}
$$

$k_0 = \omega/c_0$: isentropic wave number

Variational problem

$$\text{Find } p \in V \text{ such that}$$
$$a(q,p) = \ell(q) \qquad \forall q \in V,$$

where

$$a(q,p) = \int_{\Omega} \nabla q \cdot \nabla p - k_0^2 \int_{\Omega} qp + \mathrm{i}k_0 \int_{\Gamma_{\mathrm{io}}} qp + \frac{\mathrm{i}-1}{2}\Big(\delta_V \int_{\Gamma_{\mathrm{w}}} \nabla_{\mathsf{T}} q \cdot \nabla_{\mathsf{T}} p + k_0^2(\gamma-1)\,\delta_T \int_{\Gamma_{\mathrm{w}}} qp\Big)$$

$$\ell(q) = 2\mathrm{i}k_0 \int_{\Gamma_{\mathrm{io}}} qg$$

Norm:

$$\|p\|_W^2 = \int_{\Omega} |\nabla p|^2 + k_0^2 \int_{\Omega} |p|^2 + \delta_V \int_{\Gamma_{\mathrm{w}}} |\nabla_{\mathsf{T}} p|^2 + \delta_T(\gamma-1)k_0^2 \int_{\Gamma_{\mathrm{w}}} |p|^2$$

Solution space W: closure of $\mathscr{C}^1(\overline{\Omega})$ in $\|\cdot\|_W$

Well-posedness

Surprisingly small changes from “normal” Helmholtz theory

Lemma (Coercivity)

For any $p \in W$*,*

$$\left| a(\bar{p}, p) + 2k_0 \| p \|^2_{L^2(\Omega)} \right| \geq \frac{1}{2\sqrt{13}} \| p \|^2_W$$

Lemma (Injectivity)

For each $k_0 > 0$*, if* $p \in W$ *such that*

$$a(q, p) = 0 \qquad \forall q \in W,$$

then $p \equiv 0$.

Here we use the radiation condition on Γ_{io}

Well-posedness, finite-element approximation

- Variational problem is well posed for each $k_{\mathbf{0}} > 0$.
  (Fredholm theory)
- Well-posedness shown in the norm on W involving tangential gradients on Γ_{w}
- Finite element approximations using standard elements (continuous, piecewise polynomials) are conforming in W

Implementation

- Software like Comsol, FEniCS:
  - Specify the integrands in the variational form
  - Software assembles the system matrix
- Example (Comsol):
  - Expression $-k_0^2 qp + \nabla q \cdot \nabla p$ in integral over Ω:

```
-k0*k0*test(p)*p+test(px)*px+test(py)*py+test(pz)*pz
```

  - Expression $\nabla_\mathrm{T} q \cdot \nabla_\mathrm{T} p$ in integral over Γ_w:

```
test(pTx)*pTx+test(pTy)*pTy+test(pTz)*pTz
```

Comparing with other boundary-layer approaches

- Searby et al. (2008) suggest a post-processing approach to compute boundary losses in cavity problems:
  1. Calculate pressure field by isentropic analysis (Helmholtz equations)
  2. Use the pressure and the tangential pressure gradients at walls to compute total power loss
- No effect of phase shifts taken into account
- Their expressions for viscous & thermal losses agrees with ours:

$$P_{\text{loss}} = \frac{\delta_V}{4\omega\rho_0}\int_{\Gamma_\text{w}} |\nabla_\text{T} p|^2 + (\gamma - 1)\frac{\delta_T \omega}{4\rho_0 c^2}\int_{\Gamma_\text{w}} |p|^2,$$

- Bossart, Joly, Bruneau (2003) suggest a iterative approach (predictor–corrector) to account for boundary-layer effects
- Our approach is strongly coupled

Schmidt/Thöns–Zueva/Joly and Cremer/Pierce models

- The viscous (but not thermal) part of the BC previously derived by Schmidt, Thöns–Zueva (2014, technical report)
- A. Pierce in *Acoustics* (1981) derives a condition equivalent to our

$$v_{\mathsf{W}} = -\delta_V \frac{\mathrm{i}-1}{2} \nabla_{\mathsf{T}} \cdot \boldsymbol{u}^{\infty} - \delta_T \frac{\omega(\gamma-1)(1+\mathrm{i})}{2\gamma p_0} p^{\infty}$$

- Based on work by L. Cremer (1948)
- Appears not to have been used in numerical computations

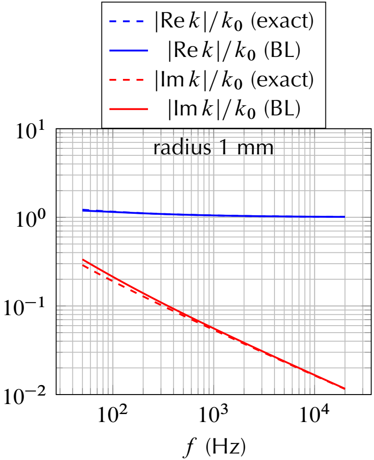
Comparing with waveguide solutions

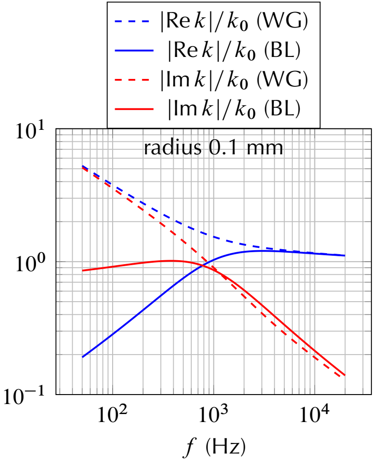
- The waveguide case extensively covered in the literature (e.g. Kirchhoff (1868); Keith (1975); Rienstra & Hirschberg (2015))
- These are exact solutions of the linearized Navier–Stokes equations in e.g. infinite tubes
- Long thin cylindrical wave guide; cross section area A, circumference L
- 1D solution ansatz: $p(z) = \hat{p}\mathrm{e}^{\mathrm{i}kz}$, $k \in \mathbb{C}$ (no transversal or circumferential dependence)
- Substituting ansatz into our variational form yields dispersion relation ($k_0 = \omega/c_0$)

$$\frac{k^2}{k_0^2} = \frac{A - \frac{\mathrm{i}-1}{2}(\gamma - 1)\delta_T L}{A + \frac{\mathrm{i}-1}{2}\delta_V L}$$

- Agrees, in the large-radius limit, with the one obtained from the exact solution

Limits of applicability





Circular tube, diameter 1 mm (solid) & 0.1 mm (dashed)

Red: exact wave number

Blue: boundary layer approximation

Numerical tests: the compression driver

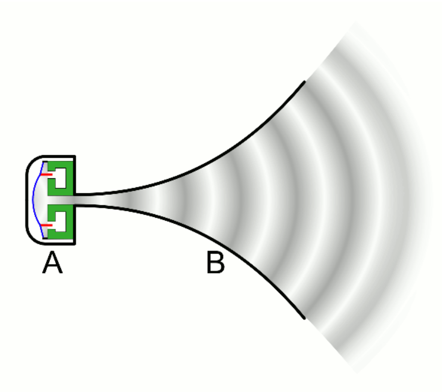


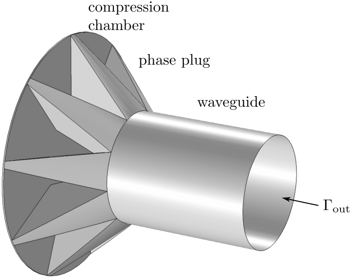
Illustration by Chetvorno, licence CCO 1.0 (Wikipedia)

- Sound source for acoustic horns
- Acoustic transformer: high pressure/low velocity → low pressure/high velocity
- Greatly improves radiation efficiency
- Contains narrow chambers, channels
- Visco–thermal losses significant

More realistic compression driver

Simplified geometry but typical dimensions:

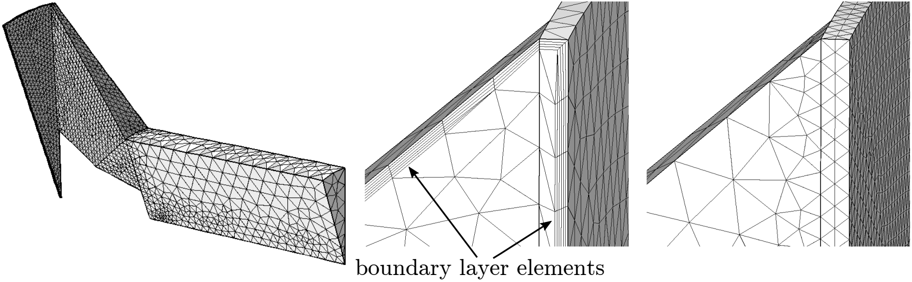
- Membrane diameter: 84 mm
- Compression chamber depth: 0.5 mm
- Compression ratio: 12



- Boundary-layer effects significant in compression chamber and phase plug
- Comparing:
    - Hybrid solver: N–S (compression chamber + phase plug) and Helmholtz (waveguide)
    - Helmholtz with our visco–thermal BC in compression chamber + phase plug

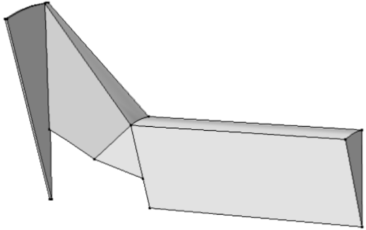
Meshes

Exploiting symmetry: computing a 20° slice



- Middle: highly stretched boundary-layer elements used for N–S case
- Right: mesh for our model

Test problem

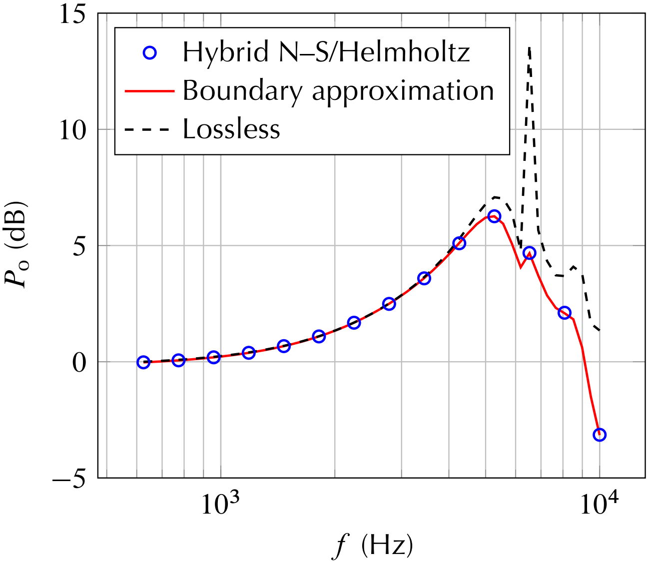


- Left boundary: stiff piston sound source; constant velocity on boundary
- Compression chamber, phase plug:
  1. Compressible N–S, vanishing velocity and temperature on boundaries, boundary-layer meshes. P^2 elements for p; P^3 for $\boldsymbol{u}$, T
  2. Helmholtz equation. BC either $\partial p/\partial n = 0$ or our proposed condition. P^2 elements
- Waveguide: Helmholtz equation, $\partial p/\partial n = 0$ BC
- Right boundary Γ_{out}: 1st-order absorbing BC

Observing radiated power

$$P_{\text{o}} = \frac{1}{2\rho_0 c_0} \int_{\Gamma_{\text{out}}} |p|^2$$

Radiated power



Computational cost

	Degrees of freedom	Memory used	Solution time per frequency
Hybrid N–S/Helmholtz	1 033 276	101 613 MB	2 111 s
Helmholtz our BC	63 725	1 242 MB	12 s
Quotient	16.21	81.8	180

- About two order of magnitude less memory and CPU time with proposed approach
- Also:
    - Our model easily solved on a laptop
    - Hybrid N–S/Helmholtz required all 24 available cores of a node in an HPC cluster

Final remarks

- Further details in:

  M. Berggren, A. Bernland, and D. Noreland. Acoustic boundary layers as boundary conditions. *J. Comput. Phys.*, 371:633–650, 2018

- The method is general, simple to implement, and seems accurate!
- Applicable for design optimization of e.g. compression drivers
- More careful look at wall curvature effects needed
- Unclear how to treat edges and corners. Boundary layers of boundary layers? Nonlinear effects?
- Taking into account wall roughness, patterns on wall, perforations?