Nonlinear stability of steady flow of Giesekus viscoelastic fluid

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Table of contents

Introduction

Elastic turbulence

Flows of viscoelastic fluids exhibit a phenomenon called **elastic turbulence**.[1]

As opposed to regular viscous fluids the flow of a viscoelastic fluid can become unstable at very low values of **Reynolds number**.

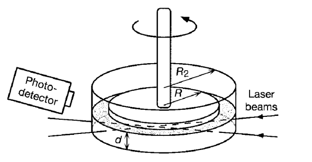

Figure 1: Experimental set-up.[1]

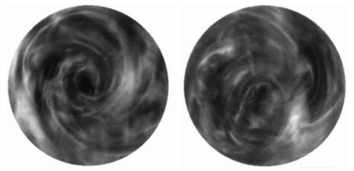

Figure 2: Elastic turbulence. $\mathrm{Wi} = 13$, $\mathrm{Re} = 0.7$.[1]

[1] Groisman and Steinberg (2000)

linear stability analysis

- linearization of the governing eqs with respect to perturbations
- yields a sufficient condition for the instability of the steady flow

nonlinear stability analysis

- energy method, Lyapunov functional
- yields a sufficient condition for the stability of the steady flow

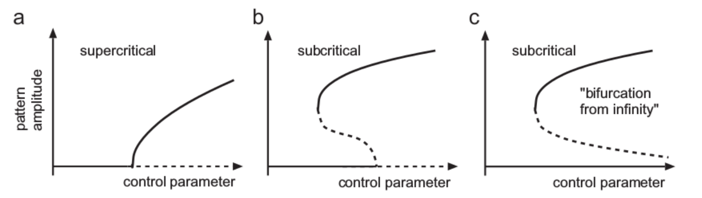

Figure 3: Bifurcation diagrams.[2]

[2] Morozov and van Saarloos (2007)

Giesekus model

Free energy and entropy production

Consider a *homogeneous incompressible* viscoelastic material characterized by the **free energy** ψ,

$$\psi \stackrel{\text{def}}{=} -c_{\mathrm{V}}\theta \left[\ln\left(\frac{\theta}{\theta_{\mathrm{ref}}}\right) - 1\right] + \frac{\mu}{2\rho}\left(\operatorname{Tr}\mathbb{B}_{\kappa_{p(t)}} - 3 - \ln\det\mathbb{B}_{\kappa_{p(t)}}\right), \tag{1}$$

and the **entropy production** $\xi = \frac{\zeta}{\theta}$, where

$$\zeta \stackrel{\text{def}}{=} 2\nu\mathbb{D}_\delta : \mathbb{D}_\delta + \kappa\frac{|\nabla\theta|^2}{\theta} + \frac{\mu^2}{2\nu_1}\operatorname{Tr}\left[\alpha\mathbb{B}^2_{\kappa_{p(t)}} + (1-3\alpha)\mathbb{B}_{\kappa_{p(t)}} + (1-\alpha)\mathbb{B}^{-1}_{\kappa_{p(t)}} + (3\alpha-2)\mathbb{I}\right]. \tag{2}$$

Governing equations

The postulated free energy and entropy production yield the governing equations for the **mechanical evolution** of the material (a similar derivation can be found in[3])

$$\operatorname{div} \mathbf{v} = 0, \tag{3a}$$

$$\rho \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \operatorname{div} \mathbb{T}, \tag{3b}$$

$$\nu_1 \overset{\triangledown}{\overline{\mathbb{B}_{\kappa_{p(t)}}}} = -\mu \left[\alpha \mathbb{B}^2_{\kappa_{p(t)}} + (1 - 2\alpha)\mathbb{B}_{\kappa_{p(t)}} - (1-\alpha)\mathbb{I} \right], \tag{3c}$$

where the Cauchy stress tensor $\mathbb{T}$ is given by the formulae [4]

$$\mathbb{T} = m\mathbb{I} + \mathbb{T}_\delta, \qquad \mathbb{T}_\delta = 2\nu\mathbb{D}_\delta + \mu\left(\mathbb{B}_{\kappa_{p(t)}}\right)_\delta .$$

Boundary conditions: $\mathbf{v} \cdot \mathbf{n}|_{\partial\Omega} = 0$, $(\mathbb{I} - \mathbf{n} \otimes \mathbf{n})\mathbf{v}|_{\partial\Omega} = \mathbf{v}_{\mathrm{bdr}}$.

[3] Hron et al. (2017)

[4] $\mathbb{D} \stackrel{\text{def}}{=} \frac{1}{2}(\nabla \mathbf{v} + \nabla \mathbf{v}^\top)$, $\mathbb{A}_\delta \stackrel{\text{def}}{=} \mathbb{A} - \frac{1}{3}(\operatorname{Tr} \mathbb{A})\mathbb{I}$, $\overset{\triangledown}{\mathbb{A}} \stackrel{\text{def}}{=} \frac{\partial \mathbb{A}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbb{A} - \nabla \mathbf{v}\mathbb{A} - \mathbb{A}\nabla \mathbf{v}^\top$

Steady flow & perturbed flow

The triplet of unknown fields is denoted by

$$\mathsf{W} \stackrel{\text{def}}{=} [\mathbf{v}, m, \mathbb{B}_{\kappa_{p(t)}}]. \tag{4}$$

The **steady flow** is denoted by

$$\widehat{\mathsf{W}} \stackrel{\text{def}}{=} [\widehat{\mathbf{v}}, \widehat{m}, \widehat{\mathbb{B}_{\kappa_{p(t)}}}], \tag{5}$$

and the **perturbation** from the steady flow is denoted by

$$\widetilde{\mathsf{W}} \stackrel{\text{def}}{=} [\widetilde{\mathbf{v}}, \widetilde{m}, \widetilde{\mathbb{B}_{\kappa_{p(t)}}}]. \tag{6}$$

NOTE The steady flow $\widehat{\mathsf{W}}$ and the perturbed flow $\widehat{\mathsf{W}} + \widetilde{\mathsf{W}}$ both represent solutions of the governing equations.

Under which conditions do we get

$$\widetilde{\mathsf{W}} \xrightarrow{t \to +\infty} 0? \tag{7}$$

Lyapunov functional

Construction of Lyapunov functional

Following the approach proposed by[5] we define the Lyapunov functional as

$$\mathcal{V}(\widetilde{\mathsf{W}} \| \widehat{\mathsf{W}}) \stackrel{\mathrm{def}}{=} E_{\mathrm{mech}}(\widehat{\mathsf{W}} + \widetilde{\mathsf{W}}) - E_{\mathrm{mech}}(\widehat{\mathsf{W}}) - \mathrm{D}_{\widehat{\mathsf{W}}} E_{\mathrm{mech}}(\widehat{\mathsf{W}})[\widetilde{\mathsf{W}}], \tag{8}$$

where

$$E_{\mathrm{mech}}(\mathsf{W}) \stackrel{\mathrm{def}}{=} \int_{\Omega} \left[\frac{1}{2} \rho \left| \mathbf{v} \right|^2 + \frac{1}{2} \mu \left(\operatorname{Tr} \mathbb{B}_{\kappa_{p(t)}} - 3 - \ln \det \mathbb{B}_{\kappa_{p(t)}} \right) \right] \mathrm{dv}, \tag{9}$$

The explicit formula for the Lyapunov functional then reads

$$\boxed{\begin{aligned} &\mathcal{V}(\widetilde{\mathsf{W}} \| \widehat{\mathsf{W}}) = \\ &\quad \int_{\Omega} \frac{1}{2} \rho |\widetilde{\mathbf{v}}|^2 \mathrm{dv} - \int_{\Omega} \frac{1}{2} \mu \ln \det \left(\mathbb{I} + \widehat{\mathbb{B}_{\kappa_{p(t)}}}^{-1} \widetilde{\mathbb{B}_{\kappa_{p(t)}}} \right) \mathrm{dv} \\ &\qquad + \int_{\Omega} \frac{1}{2} \mu \operatorname{Tr} \left(\widehat{\mathbb{B}_{\kappa_{p(t)}}}^{-1} \widetilde{\mathbb{B}_{\kappa_{p(t)}}} \right) \mathrm{dv}. \end{aligned}} \tag{10}$$

[5] Bulíček et al. (2017)

Time derivative of Lyapunov functional

The time derivative of the Lyapunov functional can be estimated as follows

$$\frac{\mathrm{d}\mathcal{V}}{\mathrm{d}t}(\widetilde{\mathsf{W}} \| \widehat{\mathsf{W}}) \leq C_1(\widehat{\mathsf{W}}, \nu, \nu_1, \mu) \left\| \nabla \widetilde{\mathsf{v}} \right\|^2_{L^2(\Omega)} + C_2(\widehat{\mathsf{W}}, \nu, \nu_1, \mu) \left\| \widetilde{\mathbb{B}_{\kappa_{p(t)}}} \right\|^2_{L^2(\Omega)}, \tag{11}$$

which in turn yields **unconditional asymptotic stability** of the steady flow provided that

$$C_1, C_2 < 0.$$

NOTE The explicit formulae for C_1, C_2 are omitted here.

Taylor–Couette problem

Application to Taylor–Couette problem

Outline of procedure:

- **Scaling**
  dimensionless numbers Re, Wi
- **Steady flow**
  numerical solution of a BVP
- **Stability of steady flow**
  evaluation of the constants C_1, C_2

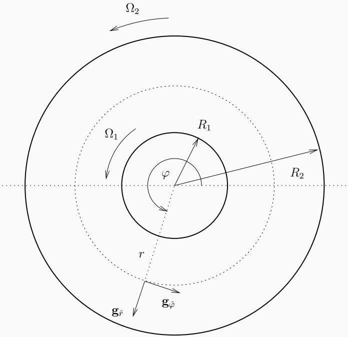

Figure 4: Taylor–Couette geometry.

Stability of steady Taylor–Couette flow

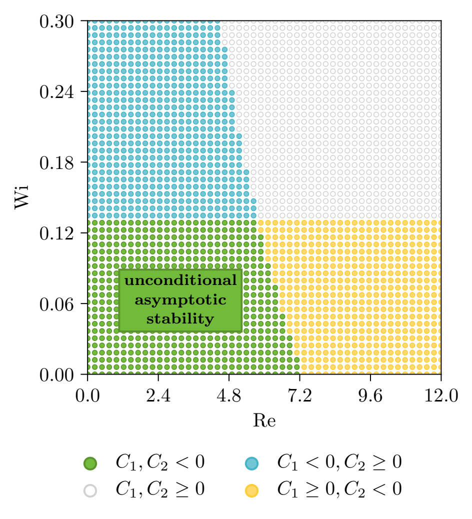

Figure 5: Stability regions in Re–Wi plane.

Conclusion

Conclusion

- We have addressed the lack of analytical results for the stability problem of flows of viscoelastic fluids by construction of a suitable **Lyapunov functional**.
- We have derived bounds on the values of Reynolds and Weissenberg numbers which guarantee the flow stability subject to any **finite perturbation**.
- We have explicitly evaluated the bounds on Reynolds and Weissenberg numbers in the case of **Taylor–Couette** type flow.

NOTE The construction of the Lyapunov functional relies on the underlying thermodynamic arguments.

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