Numerical treatment of evolutionary systems (generalised Friedrichs systems)

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Content

General class of problems

Let

- H be a Hilbert space,
- $H_\rho(\mathbb{R}; H) := \{f : \mathbb{R} \to H \ : \ f \text{ meas.}, \int_{\mathbb{R}} |f(t)|_H^2 \mathrm{e}^{-2\rho t} \, \mathrm{d}t < \infty\}$,
- inner product: $\langle f, g \rangle_\rho := \int_{\mathbb{R}} \langle f(t), g(t) \rangle_H \mathrm{e}^{-2\rho t} \, \mathrm{d}t$.

Picard: An Elementary Hilbert Space Approach to Evolutionary Partial Differential Equations, *Rend. Istit. Mat. Univ. Trieste*, 2010

Picard, McGhee: Partial Differential Equations: A unified Hilbert Space Approach, *volume 55 of De Gruyter Expositions in Mathematics*, 2011

General class of problems

Then the problem class

$$\overline{(\partial_t M_0 + M_1 + A)} U = F,$$

where

- $M_0, \ M_1$ bounded, linear, self-adjoint operators on H,
- A an unbounded, skew-selfadjoint operator on H,
- $\exists \rho_0 > 0, \ \gamma > 0 \ \forall \rho \geq \rho_0, \ x \in H: \qquad \langle (\rho M_0 + M_1) x, x \rangle_\rho \geq \gamma \langle x, x \rangle_\rho,$

has for each $\rho \geq \rho_0$ and $F \in H_\rho(\mathbb{R}, \ H)$ a unique solution $U \in H_\rho(\mathbb{R}, \ H)$ and it holds

$$|U|_\rho \leq \frac{1}{\gamma} |F|_\rho.$$

Picard: An Elementary Hilbert Space Approach to Evolutionary Partial Differential Equations, *Rend. Istit. Mat. Univ. Trieste*, 2010
Picard, McGhee: Partial Differential Equations: A unified Hilbert Space Approach, *volume 55 of De Gruyter Expositions in Mathematics*, 2011

Examples

wave equation where $f(t, \boldsymbol{x}) = 0$ for $t < 0$.

$$u_{tt} - \Delta u = f, \quad u|_{\partial\Omega} = 0, \quad u|_{t=0} = u_t|_{t=0} = 0$$

$$\left[\partial_t \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \operatorname{div} \\ \mathring{\operatorname{grad}} & 0 \end{pmatrix}\right] \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix}, \quad \partial_t F = f,\ F(0^-) = 0$$

solution space:

$$V = \left\{ (u, v) : \begin{array}{l} u \in H^1_\rho\left(\mathbb{R}_+, L^2(\Omega)\right) \cap L^2_\rho\left(\mathbb{R}_+, H^1_0(\Omega)\right), \\ v \in H^1_\rho\left(\mathbb{R}_+, L^2(\Omega)\right) \cap L^2_\rho\left(\mathbb{R}_+, H(\operatorname{div}, \Omega)\right) \end{array} \right\}$$

Examples

heat equation where $f(t, \boldsymbol{x}) = 0$ for $t < 0$.

$$u_t - \Delta u = f, \quad u|_{\partial\Omega} = 0, \quad u|_{t=0} = 0$$

$$\left[\partial_t \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \operatorname{div} \\ \mathring{\operatorname{grad}} & 0 \end{pmatrix}\right] \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}$$

solution space:

$$V = \left\{ (u, v) : \begin{array}{l} u \in H^1_\rho\left(\mathbb{R}_+, L^2(\Omega)\right) \cap L^2_\rho\left(\mathbb{R}_+, H^1_0(\Omega)\right), \\ v \in L^2_\rho\left(\mathbb{R}_+, H(\operatorname{div}, \Omega)\right) \end{array} \right\}$$

Examples

reaction-diffusion-equation where $f(t, \boldsymbol{x}) = 0$ for $t < 0$.

$$u - \Delta u = f, \quad u|_{\partial\Omega} = 0$$

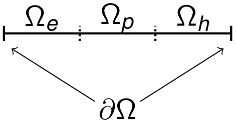
$$\left[\partial_t \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \operatorname{div} \\ \mathring{\operatorname{grad}} & 0 \end{pmatrix}\right] \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}$$

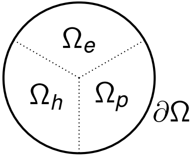
solution space:

$$V = \left\{ (u, v) : \begin{array}{l} u \in L^2_\rho\left(\mathbb{R}_+, H^1_0(\Omega)\right), \\ v \in L^2_\rho\left(\mathbb{R}_+, H(\operatorname{div}, \Omega)\right) \end{array} \right\}$$

Examples

changing type systems





$$\left[\partial_t \begin{pmatrix} \chi_{\Omega_h \cup \Omega_p} & 0 \\ 0 & \chi_{\Omega_h} \end{pmatrix} + \begin{pmatrix} \chi_{\Omega_e} & 0 \\ 0 & \chi_{\Omega_p \cup \Omega_e} \end{pmatrix} + \begin{pmatrix} 0 & \operatorname{div} \\ \mathring{\operatorname{grad}} & 0 \end{pmatrix}\right] \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}$$

solution space:

$$V = \left\{ (u, v) : \begin{array}{l} u \in H^1_\rho\left(\mathbb{R}_+, L^2(\Omega_h \cup \Omega_p)\right) \cap L^2_\rho\left(\mathbb{R}_+, H^1_0(\Omega)\right), \\ v \in H^1_\rho\left(\mathbb{R}_+, L^2(\Omega_h)\right) \cap L^2_\rho\left(\mathbb{R}_+, H(\operatorname{div}, \Omega)\right) \end{array} \right\}$$

no transmission conditions

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Time – Discontinuous Galerkin

continuous form:

Find $U \in V$, such that for all $\Phi \in V$ and $m = 1, \ldots, M$

$$\langle (\partial_t M_0 + M_1 + A)U, \Phi \rangle_{\rho,m} = \langle F, \Phi \rangle_{\rho,m},$$

where $\langle \cdot, \cdot \rangle_{\rho,m}$ is the weighted inner product localised to $I_m := (t_{m-1}, t_m]$.

Time – Discontinuous Galerkin

semidiscrete form:

Find $U^\tau \in \mathcal{V}^\tau := \mathcal{P}_q^{disc}([0,T],H)$, such that for all $\Phi \in \mathcal{V}^\tau$ and m

$$\langle (\partial_t M_0 + M_1 + A)U^\tau, \Phi\rangle_{\rho,m} + \langle M_0 [\![U^\tau]\!]_{m-1}, \Phi^+_{m-1}\rangle_H = \langle F, \Phi\rangle_{\rho,m},$$

where $[\![\cdot]\!]_{m-1}$ is the jump at the time t_{m-1}.

There is exactly one solution for each $F \in H_\rho(\mathbb{R}, H)$.

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Finite Element Method

discrete form:
Find $U_h \in \mathcal{V}_h^\tau$, such that for all $\Phi \in \mathcal{V}_h^\tau$, $m = 1, \ldots, M$

$$Q\left[(\partial_t M_0 + M_1 + A)U_h, \Phi\right]_{\rho,m} + \langle M_0 [\![U_h]\!]_{m-1}, \Phi_{m-1}^+ \rangle_H = Q\left[F, \Phi\right]_{\rho,m},$$

where $[\![\cdot]\!]_{m-1}$ is the jump at the time t_{m-1} and $Q\left[\cdot,\cdot\right]_{\rho,m}$ is a weighted right-sided Gauß-Radau quadrature rule exactly for $p \in \mathcal{P}_{2q}(I_m, H)$.
$\mathcal{V}_h^\tau$ is defined by using piecewise

- polynomials in t of degree q, globally discontinuous,
- polynomials in $\boldsymbol{x}$ of degree k, globally continuous for $u_h(t,\cdot)$, $\Rightarrow H^1$-conform
- Raviart-Thomas elements in $\boldsymbol{x}$ of degree $k-1$ for $v_h(t,\cdot)$, $\Rightarrow H(div)$-conform

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Results

Assuming

$$U \in H_\rho^1(\mathbb{R}; H^k(\Omega) \times H^k(\Omega)) \cap H_\rho^{q+2}(\mathbb{R}; L^2(\Omega) \times L^2(\Omega))$$

as well as

$$AU \in H_\rho(\mathbb{R}; H^k(\Omega) \times H^k(\Omega)),$$

we obtain

$$\mathrm{e}^{-2\rho T} \cdot \sup_{t\in[0,T]} \langle M_0(U-U_h)(t), (U-U_h)(t)\rangle_H + \|U-U_h\|_{Q,\rho}^2 \le C(\tau^{2(q+1)} + Th^{2k}).$$

Remark: High regularity only needed in the interior of each cell.

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Drawback of this approach

- No continuity in time, although jumps converge towards zero.
- Dissipative method in the sense

$$\mathrm{e}^{-2\rho t}\langle M_0 U(t), U(t)\rangle_H + 2\langle(\rho M_0 + M_1)U, U\rangle_{\rho,(0,t)} = \langle M_0 U(0), U(0)\rangle_H.$$

vs.

$$\mathrm{e}^{-2\rho t_i}\langle M_0 U^\tau_{i-}, U^\tau_{i-}\rangle_H + 2\langle(\rho M_0 + M_1)U^\tau, U^\tau\rangle_{\rho,(0,t_i)} + 2\sum_{m=1}^{i} \mathrm{e}^{-2\rho t_{m-1}}\langle M_0 [\![U^\tau]\!]^{x_0}_{m-1}, [\![U^\tau]\!]^{x_0}_{m-1}\rangle = \langle M_0 U^\tau_{0-}, U^\tau_{0-}\rangle_H.$$

Another approach: cGP – continuous-Galerkin-Petrov

semidiscrete form:
Find $U^\tau \in \mathcal{V}^\tau := \mathcal{P}_q^{cont}([0,T],H)$, such that for all $\Phi \in \mathcal{W}^\tau := \mathcal{P}_{q-1}^{disc}([0,T],H)$ and m

$$\begin{aligned} \langle(\partial_t M_0 + M_1 + A)U^\tau, \Phi\rangle_{\rho,m} &= \langle F, \Phi\rangle_{\rho,m}, \\ U(t_{m-1}^+) &= U(t_{m-1}^-). \end{aligned}$$

Is there exactly one solution for each $F \in H_\rho(\mathbb{R}, H)$?

- true for M_0 with trivial null-space,
- probably true for general M_0 and $F \in H_\rho^1(\mathbb{R}, H)$.

What about a convergence analysis?

Results for semi-discrete formulation

Let $P : C(I_m, H) \to \mathcal{P}_q(I_m, H)$ be the cGP-interpolation operator fulfilling

$$
\begin{aligned}
(PU - U)(t_{m-1}) = (PU - U)(t_m) = 0, \\
\langle PU - U, W \rangle_{\rho,m} = 0 \quad \forall W \in \mathcal{P}_{q-2}(I_m, H).
\end{aligned}
$$

Let $\eta = U - PU$ and $\xi = PU - U^\tau$. Then $U - U^\tau = \eta + \xi$ and the Galerkin orthogonality $\langle (\partial_t M_0 + M_1 + A)(U - U^\tau), \Phi \rangle_{\rho,m} = 0$ give

$$
\begin{aligned}
\langle (\partial_t M_0 + M_1 + A)\xi, \Phi \rangle_{\rho,m} &= -\langle (\partial_t M_0 + M_1 + A)\eta, \Phi \rangle_{\rho,m} \\
&= -\langle (2\rho M_0 + M_1 + A)\eta, \Phi \rangle_{\rho,m} \\
&\leq \left(\|2\rho M_0 + M_1\| \, \|\eta\|_{\rho,m} + \|A\eta\|_{\rho,m} \right) \|\Phi\|_{\rho,m}.
\end{aligned}
$$

Results for semi-discrete formulation

Let $\Phi = \Pi_{q-1}\xi$ be the L^2-projection into $\mathcal{P}_{q-1}(I_m, H)$. Then

$$\gamma\|\Pi_{q-1}\xi\|_{\rho,m}^2 + \langle M_0\xi(t), \xi(t)\rangle_H e^{-2\rho t}\Big|_{t_{m-1}}^{t_m} \leq \frac{1}{\gamma}\left(\|2\rho M_0 + M_1\|\,\|\eta\|_{\rho,m} + \|A\eta\|_{\rho,m}\right)^2$$

Let $\Phi = I_{q-1}\xi$ be the Gauß-Legendre-interpolation into $\mathcal{P}_{q-1}(I_m, H)$. Then

$$\gamma\|I_{q-1}\xi\|_{\rho,m}^2 + \langle M_0\xi(t), \xi(t)\rangle_H e^{-2\rho t}\Big|_{t_{m-1}}^{t_m} \leq \frac{1}{\gamma}\left(\|2\rho M_0 + M_1\|\,\|\eta\|_{\rho,m} + \|A\eta\|_{\rho,m}\right)^2,$$

because $I_{q-1} = \Pi_{q-1}$ on $\mathcal{P}_q(I_m, H)$.

Results for semi-discrete formulation

- Optimal results for
$$\|U - \Pi_{r-1} U^\tau\|_{\rho,[0,t_m]},$$
- Optimal results for
$$\langle M_0(U - U^\tau)(t_m), (U - U^\tau)(t_m)\rangle_H \mathrm{e}^{-2\rho t_m},$$
- If M_0 has trivial null space: Optimal results for
$$\|U - U^\tau\|_{\rho,[0,t_m]}$$
- Suboptimal results for
$$\sup_{t\in[0,t_m]} \langle M_0(U - U^\tau)(t), (U - U^\tau)(t)\rangle_H \mathrm{e}^{-2\rho t}.$$

cGP – continuous-Galerkin-Petrov

discrete form:
Find $U_h \in \mathcal{V}_h^\tau$, such that for all $\Phi \in \mathcal{W}_h^\tau$, $m = 1, \dots, M$

$$Q\left[(\partial_t M_0 + M_1 + A)U_h, \Phi\right]_{\rho,m} = Q\left[F, \Phi\right]_{\rho,m},$$

where $Q\left[\cdot,\cdot\right]_{\rho,m}$ is a weighted Gauß-Lobatto quadrature rule exactly for $p \in \mathcal{P}_{2q-1}(I_m, H)$.

Discrete spaces $\mathcal{V}_h^\tau$ and $\mathcal{W}_h^\tau$ are defined by using piecewise

- polynomials in t of degree q / $q-1$, globally continuous / discontinuous,
- polynomials in $\boldsymbol{x}$ of degree k, globally continuous for $u_h(t,\cdot)$,
- Raviart-Thomas elements in $\boldsymbol{x}$ of degree $k-1$ for $v_h(t,\cdot)$

Drawback of this approach

- Needs continuity in time everywhere, although solution theory valid also for data not supporting continuity in some subdomains.
- Needs initial conditions everywhere.

Goal:
mixed cGP/dG method in time, that is

- piecewise polynomial of order q,
- continuous, where U is guaranteed to be continuous (range of M_0),
- allows discontinuity, where U might be discontinuous (null-space of M_0).

Problem:
Definition and solvability of semi-discrete problem

1d-example

Let $\Omega = \left[-\frac{3\pi}{2}, \frac{3\pi}{2}\right]$, $\Omega_h = \left[-\frac{3\pi}{2}, 0\right]$, $\Omega_p = \left[0, \frac{3\pi}{2}\right]$ and

$$\left[\partial_t \begin{pmatrix} 1 & 0 \\ 0 & \chi_{\Omega_h} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \chi_{\Omega_p} \end{pmatrix} + \begin{pmatrix} 0 & \partial_x \\ \mathring{\partial}_x & 0 \end{pmatrix}\right] \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}$$

with homogeneous Dirichlet-conditions for u.

$$\begin{aligned} f(t,x) &= \Big((1+t-2\mathrm{e}^t)\chi_{(-\frac{\pi}{2},0)}(x) + \mathrm{e}^t\big[\chi_{(\frac{\pi}{2},\frac{3\pi}{2})} - \chi_{(0,\frac{\pi}{2})}\big](x)\Big)\cos(x) \\ &\quad + \big[\chi_{(0,\pi)} - \chi_{(\pi,\frac{3\pi}{2})}\big](x), \\ g(t,x) &= \chi_{(0,\pi)}(x)x + \chi_{(\pi,\frac{3\pi}{2})}(x)(2\pi - x) \\ &\quad - (\mathrm{e}^t - 1)\big[\chi_{(\frac{\pi}{2},\frac{3\pi}{2})} - \chi_{(0,\frac{\pi}{2})}\big](x)\sin(x). \end{aligned}$$

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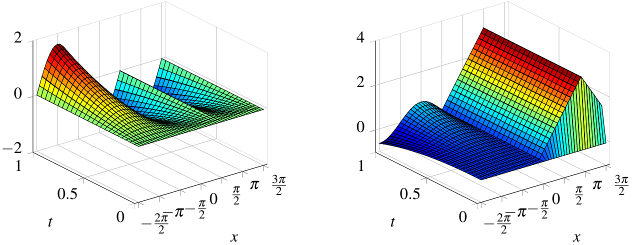
1d-example

Let $\Omega = \left[-\frac{3\pi}{2}, \frac{3\pi}{2}\right]$, $\Omega_h = \left[-\frac{3\pi}{2}, 0\right]$, $\Omega_p = \left[0, \frac{3\pi}{2}\right]$ and

$$\left[\partial_t \begin{pmatrix} 1 & 0 \\ 0 & \chi_{\Omega_h} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \chi_{\Omega_p} \end{pmatrix} + \begin{pmatrix} 0 & \partial_x \\ \mathring{\partial}_x & 0 \end{pmatrix}\right] \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}$$

with homogeneous Dirichlet-conditions for u.



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1d-example, dG-method

$N = M$	$E_{\sup}(U - U_h)$		$\|U - U_h\|_{Q,\rho}$	
		$k = 2,\ q = 1$		
96	3.577e-04		6.400e-05	
192	9.010e-05	1.99	1.601e-05	2.00
384	2.261e-05	1.99	4.002e-06	2.00
768	5.662e-06	2.00	1.001e-06	2.00
		$k = 3,\ q = 2$		
96	6.981e-08		7.500e-10	
192	8.726e-09	3.00	2.343e-11	5.00
384	1.091e-09	3.00	7.329e-13	5.00
768	1.363e-10	3.00	2.474e-14	4.89

Theory: $\min\{k, q+1\}$ for smooth U

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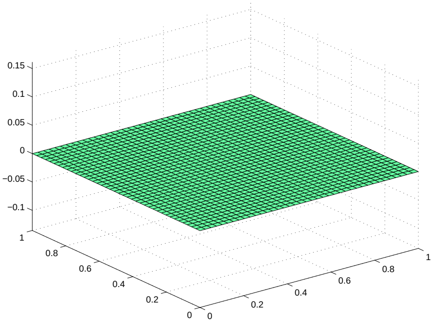
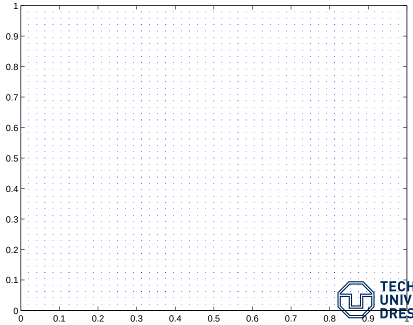
2d-example

Sei $\Omega_h = \left\{(x, y) \in \left[\frac{1}{4}, \frac{3}{4}\right]^2\right\}$, $\Omega_p = [0, 1]^2 \setminus \Omega_h$ und $\Omega_e = \emptyset$, sowie

$$f(t, \boldsymbol{x}) = 2\sin(\pi t)\chi_{\{x<0.5\}}(\boldsymbol{x})$$





2d-example, dG-method

$N = M$	$E_{\sup}(U - U_h)$		$\|U - U_h\|_{Q,\rho}$	
$k = 2$, $q = 1$				
8	6.635e-03		3.823e-03	
16	2.002e-03	1.73	1.244e-03	1.62
32	6.497e-04	1.62	4.691e-04	1.41
64	2.404e-04	1.43	2.101e-04	1.16
$k = 3$, $q = 2$				
8	1.337e-03		9.640e-04	
16	4.496e-04	1.57	3.962e-04	1.28
32	1.851e-04	1.28	1.904e-04	1.06
64	7.992e-05	1.21	9.710e-05	0.97

Theory: $\min\{k, q+1\}$ for smooth U

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Thank you for your attention!