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IMPLEMENTATION OF FUNCTIONAL-TYPE A POSTERIORI ERROR ESTIMATES FOR LINEAR PROBLEMS IN SOLID MECHANICS

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joint work with

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Problems in 2D

- Classical Elasticity
- Cosserat Elasticity (microstructure)
- **Reissner-Mindlin Plates**

A priori and a posteriori estimates

- u – exact solution.
- u_h – approximate solution (h – a mesh parameter).
- $\|u - u_h\| \leq C(u, \ldots)h^{\alpha}$, $\alpha > 0$ – a priori error estimate.
- $\|u - u_h\| \leq \mathcal{M}(u_h, \ldots)$ – a posteriori error estimate.
- Efficiency index $I_{eff} = \mathcal{M} / \|u - u_h\|$.

A posteriori error estimates

1. Error estimates in global norms
   Indicators
   Guaranteed and reliable estimates

2. Adaptive algorithms
   Construction of local indicators
   Convergence of adaptive algorithms

Key features of classical approaches to error control in FEM

- Element u_h – the exact solution of a discrete problem (Galerkin approximation).
- Simplicity of approaches and low computational costs are preferable by engineers.
- In most cases, indicators are considered rather than majorants (reliable functionals).

It is better to have an estimate that is efficiently applicable, but reliable and robust to deviations.

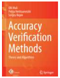
Functional approach: General references

Repin S.,Xanthis L.S. *Comp.Meth.Appl.Mech.Engrg.* 1996

Repin S. *Math. Comput.* 2000

Neittaanmäki P., Repin S. 2004

Repin S. 2008



Mali O., Neittaanmäki P., Repin S. 2014

Error estimation

Constitutive Relation, Residual-Based, Postprocessing (SPR,...),...

- Boisse P., Perrin S., Coffignal G., Hadjeb K. 1999
- Liberman E. 2001
- Carstensen C. 2002, Carstensen C., Weinberg K. 2001, 2003, Carstensen C., Schöberl J. 2006, Carstensen C. et al. 2011, Carstensen C., Hu J. 2008, Hu J., Huang Y. 2010
- Boroomand B., Ghaffarian M., Zienkiewicz O.C. 2004, Castellazzi G., de Miranda S., Ubertini F. 2011
- Lovadina C., Stenberg R. 2005, Beirão da Veiga L., Chinosi C., Lovadina C., Stenberg R. 2008, Beirão da Veiga L., Niiranen J., Stenberg R. 2013
- ...

Functional approach: KL and RM plates

- Neittaanmäki P., Repin S. *East-West J. Numer. Math.* 2001
- Frolov M., Neittaanmäki P., Repin S. *ECCOMAS* 2004
- Frolov M. *Russ. J. Numer. Anal. Math. Model.* 2004

- Repin S., Frolov M. *Zap. Nauchn. Sem. POMI* 2004 (J. of Math. Sci., 2006)
- Frolov M., Neittaanmäki P., Repin S. *Probl. Mat. Anal.* 2005 (J. of Math. Sci., 2006)

- Frolov M. *Mesh Methods for Boundary-Value Problems and Applications*, Kazan, 2014
- Frolov M., Chistiakova O. *IOP Conf. Series: Materials Science and Engineering* 208 (2017) 012043

Reissner-Mindlin plates

This model describes a bending of linearly elastic plates of small to moderate thickness in terms of the pair of variables in $\Omega \in \mathbb{R}^2$
$u = u(x)$ — scalar-valued function (displacement), $\theta = \theta(x)$ — vector-valued function (rotations)

$$\begin{cases} -\mathbf{Div}\,(\mathrm{C}\varepsilon(\theta)) = \gamma & \text{in } \Omega, \\ -\mathbf{div}\gamma = g & \text{in } \Omega, \\ \gamma = \lambda t^{-2}(\nabla u - \theta) & \text{in } \Omega, \end{cases} \tag{1}$$

t — thickness of a plate; $\lambda = \frac{Ek}{2(1+\nu)}, \varepsilon(\theta) = \frac{1}{2}(\nabla\theta + (\nabla\theta)^T)$;
gt^3 represents the transverse loading;
C — is the tensor of bending moduli;
E and ν — material constants;
k — a correction factor (5/6).

Boundary conditions

- Γ_{D} and Γ_{S} are two non-intersecting parts of the boundary Γ;
- Γ_{D} — clamped part,
  $u = 0, \theta = 0$ on Γ_{D};
- Γ_{S} — free part,
  $\partial u / \partial n = \theta \cdot n$ on Γ_{S},
  $\mathrm{C}\varepsilon(\theta)n = 0$ on Γ_{S};

  Γ_{S} — supported part
  instead of $\partial u / \partial n = \theta \cdot n$ on Γ_{S}, one has $u = 0$ on Γ_{S}.

The case $\Gamma = \Gamma_{\mathrm{D}}$ was considered in 2004-2005.

Weak formulation

Find a triple $(u, \theta, \gamma) \in U \times \Theta \times Q$

$$
\begin{cases}
\int\limits_{\Omega} \mathbb{C}\varepsilon(\theta) : \varepsilon(\varphi)\, d\Omega - \int\limits_{\Omega} \gamma \cdot \varphi \, d\Omega = 0, & \forall \varphi \in \Theta, \\
\int\limits_{\Omega} \gamma \cdot \nabla w \, d\Omega = \int\limits_{\Omega} g w \, d\Omega, & \forall w \in U, \\
\int\limits_{\Omega} \left(\lambda^{-1} t^2 \gamma - (\nabla u - \theta)\right) \cdot \tau \, d\Omega = 0, & \forall \tau \in Q,
\end{cases} \tag{2}
$$

$$
U = \left\{ w \in \mathbb{W}_2^1(\Omega) \mid w = 0 \text{ on } \Gamma_{\mathrm{D}} \right\},
$$
$$
\Theta = \left\{ \varphi \in \mathbb{W}_2^1(\Omega, \mathbb{R}^2) \mid \varphi = 0 \text{ on } \Gamma_{\mathrm{D}} \right\},
$$
$$
Q = \mathbb{L}_2(\Omega, \mathbb{R}^2).
$$

Assumptions

Domain $\Omega \subset \mathbb{R}^2$ – bounded connected domain with Lipschitz-continuous boundary,
$g \in \mathbb{L}_2(\Omega)$,
C — symmetric and there exist ξ_1 and ξ_2

$$\xi_1 |\varkappa|^2 \leq \mathrm{C}\varkappa : \varkappa \leq \xi_2 |\varkappa|^2 \quad \forall \varkappa \in \mathbb{M}^{2\times 2}_{sym}, \quad |\varkappa|^2 = \varkappa : \varkappa,$$

where $\mathbb{M}^{2\times 2}_{sym}$ — the space of symmetric tensors of 2nd rank.
For an isotropic material

$$\mathrm{C}\varkappa = \frac{E}{12(1-\nu^2)} \left((1-\nu)\varkappa + \nu \, \mathrm{tr} \varkappa \, \mathbb{I} \right),$$

where $\mathbb{I}$ — identity tensor of 2nd rank.

Elements of an error estimate

$(\tilde{u}, \tilde{\theta})$ — an arbitrary pair of conforming approximations,
$e_{\tilde{u}} = u - \tilde{u}$, $e_{\tilde{\theta}} = \theta - \tilde{\theta}$, $e_{\tilde{\gamma}} = \gamma - \tilde{\gamma}$ — errors.

We introduce **free** elements

1. vector $\tilde{\tilde{y}}$
2. arbitrary tensor $\tilde{\tilde{\varkappa}}$ divided into two parts

$\tilde{\tilde{\varkappa}} = [\tilde{\tilde{\varkappa}}^1,\ \tilde{\tilde{\varkappa}}^2]$, where

$$\tilde{\tilde{y}},\quad \tilde{\tilde{\varkappa}}^1,\quad \tilde{\tilde{\varkappa}}^2 \quad \in \quad \mathbb{H}(\Omega, \mathbf{div}) := \left\{ y \in \mathbb{L}_2(\Omega, \mathbb{R}^2) \mid \mathbf{div} y \in \mathbb{L}_2(\Omega) \right\}.$$

Auxiliary notations and inequalities

$$\|e_{\tilde{\gamma}}\|_{\Omega}^2 := \int\limits_{\Omega} |e_{\tilde{\gamma}}|^2 \, dx, \qquad \|e_{\tilde{\theta}}\|^2 := \int\limits_{\Omega} \mathbb{C}\varepsilon(e_{\tilde{\theta}}) : \varepsilon(e_{\tilde{\theta}}) \, dx$$

$$\|\nabla \varphi\|_{\Omega}^2 \leq \mathfrak{c}_I^2 \|\varphi\|^2$$

$$\|\varphi\|_{\Omega}^2 \leq \mathfrak{c}_{II}^2 \|\varphi\|^2$$

$$\frac{1}{|\Omega|} \|w\|_{\Omega}^2 + \frac{1}{|\Gamma_{\mathrm{S}}|} \|w\|_{\Gamma_{\mathrm{S}}}^2 \leq \mathfrak{c}_{III}^2 \|\nabla w\|_{\Omega}^2$$

$$\frac{1}{|\Omega|} \|\varphi\|_{\Omega}^2 + \frac{1}{|\Gamma_{\mathrm{S}}|} \|\varphi\|_{\Gamma_{\mathrm{S}}}^2 \leq \mathfrak{c}_{IV}^2 \|\varphi\|^2$$

Functional-type error majorant

$$\|e_{\tilde\theta}\|^2 + \lambda^{-1} t^2 \|e_{\tilde\gamma}\|^2_\Omega \leq \hat{a}^2 + \lambda^{-1} t^2 \hat{b}^2 \qquad (3)$$

$$\begin{aligned}\hat{a} = {} & \|\mathrm{C}^{-1}\mathbf{sym}(\tilde{\tilde{\varkappa}}) - \varepsilon(\tilde\theta)\| + \mathfrak{c}_I \|\mathbf{skew}(\tilde{\tilde{\varkappa}})\|_\Omega + \\ & + \mathfrak{c}_{II}\mathfrak{c}_{III}\sqrt{|\Omega| \;\; \|g + \mathbf{div}\tilde{\tilde{y}}\|^2_\Omega \; + \; |\Gamma_\mathrm{S}| \;\; \|\tilde{\tilde{y}} \cdot n\|^2_{\Gamma_\mathrm{S}}} + \\ & + \mathfrak{c}_{IV}\sqrt{|\Omega| \;\; \|\tilde{\tilde{y}} + [\mathbf{div}\tilde{\tilde{\varkappa}}^1, \mathbf{div}\tilde{\tilde{\varkappa}}^2]\|^2_\Omega \; + \; |\Gamma_\mathrm{S}| \;\; \|[\tilde{\tilde{\varkappa}}^1 \cdot n, \tilde{\tilde{\varkappa}}^2 \cdot n]\|^2_{\Gamma_\mathrm{S}}},\end{aligned}$$

$$\hat{b} = \|\tilde{\tilde{y}} - \tilde\gamma\|_\Omega + \mathfrak{c}_{III}\sqrt{|\Omega| \;\; \|g + \mathbf{div}\tilde{\tilde{y}}\|^2_\Omega \; + \; |\Gamma_\mathrm{S}| \;\; \|\tilde{\tilde{y}} \cdot n\|^2_{\Gamma_\mathrm{S}}}.$$

The estimate **is exact** for $\tilde{\tilde{\varkappa}} = \mathrm{C}\varepsilon(\theta)$ and $\tilde{\tilde{y}} = \gamma$

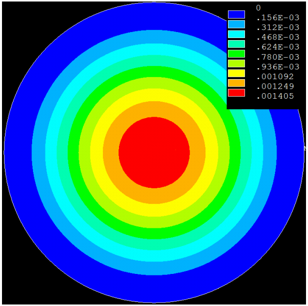
Commercial software

- Hidden implementations.
- Algorithms behave different from version to version.
- ANSYS Mechanical: variety of elements (SHELL, SOLID,...).
- Incorrect settings? Locking? Bugs?

Bathe K.-J., Dvorkin E.N. *Int. J. Numer. Methods Eng.* 1986,
Simo J.C., Armero F. *Int. J. Numer. Methods Eng.* 1992.

- Can the functional approach provide reliable and efficient error control even on coarse meshes?
- Is this approach practically applicable for non-Galerkin approximations as well as for Galerkin ones?

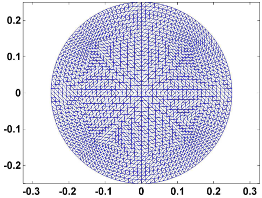
Example I

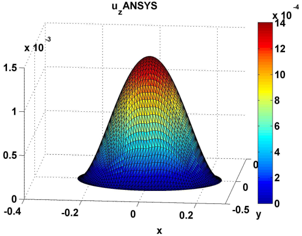


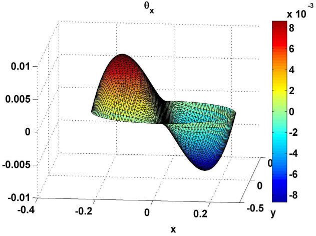
radius 0.25 m
thickness 0.0025 m
$E = 2e11\ \mathrm{N/m^2}$
$\nu = 0.3$

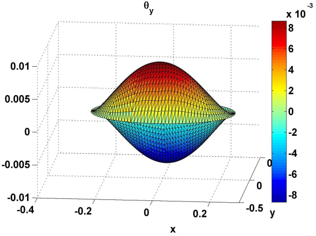
ANSYS Verification Manual: VM138, S.P. Timoshenko
SHELL181-ELEMENT, Loading $= 6585.175\ \mathrm{N/m^2}$

Example I. ANSYS solution









Example I. Error estimation: Results

*Locking for Galerkin approximation: maximum of u is 0.027e-2 instead of 0.140e-2

DOFs	123	435	1635	6339*
Error	0.689e5	0.359e5	0.181e5	0.842e4
Estimate	0.101e6	0.527e5	0.266e5	0.133e5
ratio	1.5	1.5	1.5	1.5

Conclusions

- The functional approach is suitable even for coarse meshes
- It is robust due to unknown information about numerical procedures

- The approach forms a good independent tool for error control for commercial software

Remarks and further tasks

- Task 1: **Construction of adaptive algorithms**
- Task 2: Complete theoretical justification (efficiency estimates)
- Task 3: Application of new approximations

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Solve-Estimate-Mark-Refine

1. S: compute $\tilde{u}$ on a current finite element mesh;
2. E: compute estimate from individual loads to elements;
3. M: mark elements of a mesh with large local errors by some marking strategy;
4. R: divide marked elements and locally refine a mesh.

Solver:

Bathe K.-J., Dvorkin E.N. *Int. J. Numer. Methods Eng.* 1985

Estimator:

Functional a posteriori error estimate, 2014

Marker:

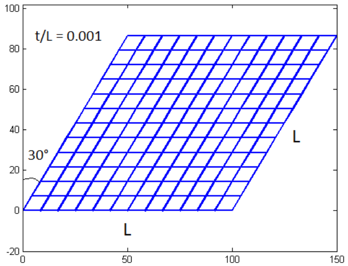
Local error indicator larger than prescribed threshold

Refiner:

Karavaev A.S., Kopysov S.P. *Bulletin of Udmurt University* 2013

Example II

Carstensen C. et al. *Comput. Methods Appl. Mech. Engrg.* 2011
Razzaque plate

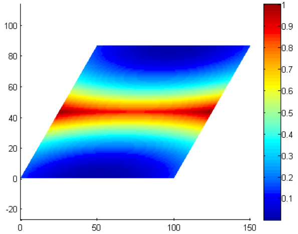


side 1.0 m
thickness 0.1 m
$E = 1092\ \mathrm{N/m^2}$
$\nu = 0.3$

Uniform loading
Horizontal edges are hard clamped

Example II. Solver and a posteriori estimate

Local error indicator:

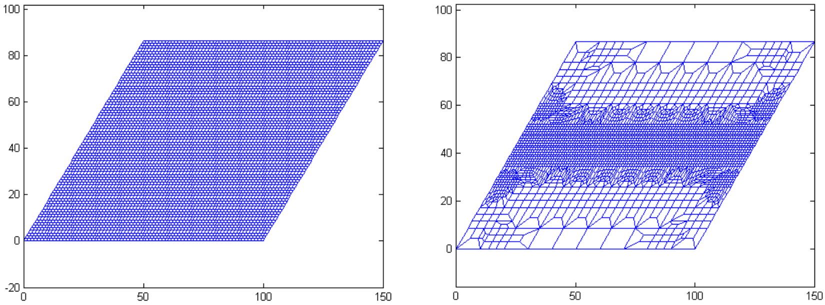


Solver and estimate comparison:

Mesh	4x4		8x8		16x16		32x32	
	C.	Ours	C.	Ours	C.	Ours	C.	Ours
Disp.	0.67	0.64	0.76	0.75	0.78	0.78	0.79	0.79
Err.	16.1	18.3	4.7	4.9	1.7	1.4	0.9	0.9
I_{eff}	0.35*	1.8	0.87*	1.8	0.97*	1.8	0.99*	1.8

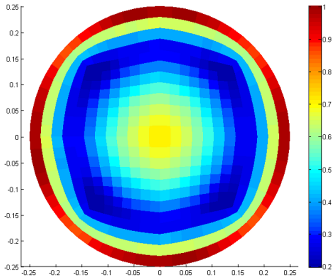
Example II. Refinement: Uniform vs Adaptive

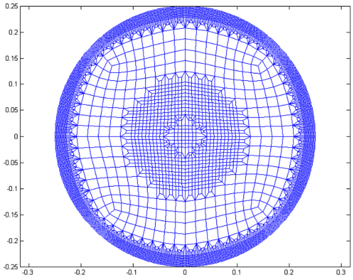
Number of elements: 5174 vs 3076
Relative error: 0.62% vs 0.48%



Back to Example I

Local error indicator and mesh adaptation:





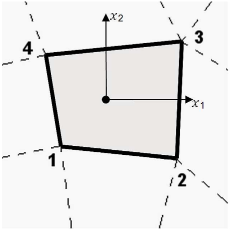
Classical and Cosserat elasticity: a posteriori error control

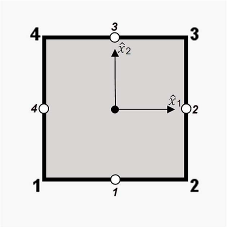
- Repin S. *de Gruyter* 2008 (non-symmetric),
  Frolov M. *Comp. Math. Math. Phys.* 2013, ...
- Repin S., Frolov M. *Probl. Math. Anal.* 2011,
  Frolov M. *J. Appl. Math. Mech.* 2014,
  RMMM, *ENUMATH*-2013, *ACOMEN*-2014,
  Churilova M., Frolov M. *Mater. Phys. Mech.* 2017,...

Conclusions:
explicit symmetry condition for tensor variable + standard FEM approximations = unsatisfactory results
implicit symmetry condition + FE for mixed methods = reliable estimates with efficient implementation

Raviart–Thomas approximation

- P.A. Raviart, J.M. Thomas. *Lecture Notes in Mathematics.* Berlin: Springer, 1977





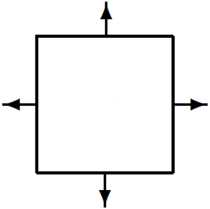
Mapping of the reference square $\hat{\mathcal{K}}$ to arbitrary quadrilateral (DOFs – normal components of a vector-field in the midpoints)

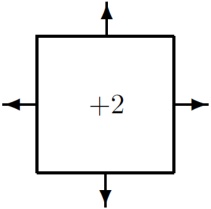
$\mathcal{RT}_0(\hat{\mathcal{K}}) = \mathcal{P}_{1,0}(\hat{\mathcal{K}}) \times \mathcal{P}_{0,1}(\hat{\mathcal{K}})$, where $\mathcal{P}_{i,j}(\hat{\mathcal{K}})$ – space of polynomials over $\hat{\mathcal{K}} = (-1,1) \times (-1,1)$ power of i or less on $\hat{x}_1$ and j – on $\hat{x}_2$.

Arnold-Boffi-Falk approximation

- D.N. Arnold, D.Boffi, R.S. Falk. *SIAM J. Numer. Anal.*, 2005

$\mathcal{ABF}_0(\hat{\mathcal{K}}) = \mathcal{P}_{2,0}(\hat{\mathcal{K}}) \times \mathcal{P}_{0,2}(\hat{\mathcal{K}})$ – a wider space that includes the original DOFs on edges and two additional internal DOFs on every element





(1) $\mathcal{RT}_0(\hat{\mathcal{K}})$ (2) $\mathcal{ABF}_0(\hat{\mathcal{K}})$

DOFs for elements under consideration

THANK YOU FOR ATTENTION

DEAR SERGEY IGOREVICH

HAPPY BIRTHDAY

WITH WARMEST REGARDS
FROM ALMA MATER