On the well-posedness of Galbrun's equation

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Derivation of Galbrun's equation—Euler's equations

Homentropic flow

$$\rho D u + \nabla p = \rho \varphi \qquad \text{Momentum conservation}$$
$$D\rho + \rho \nabla \cdot u = 0 \qquad \text{Mass conservation}$$
$$p = \Sigma(\rho) \qquad \text{Equation of state}$$

- Nonlinear conservation laws
  - u velocity
  - ρ (mass) density
  - p pressure
  - φ force density
  - $D = \partial_t + u \cdot \nabla$ material derivative
- Thermodynamic state equation
  - Speed of sound: $c = \sqrt{\Sigma'(\rho)} \approx 340\,{}^{\mathrm{m}}/_{\mathrm{s}}$ in air

Derivation of Galbrun's equation—Linearization

- Linearization ansatz ($\phi \in \{u, \rho, p, \varphi\}$)
  - $\phi(x,t) = \phi_0(x,t) + \delta\phi(x,t)$

$$\rho_0 D_0 u_0 + \nabla p_0 = \rho_0 \varphi_0 \qquad \text{Momentum conservation}$$

$$D_0 \rho_0 + \rho_0 \nabla \cdot u_0 = 0 \qquad \text{Mass conservation}$$

$$p_0 = \Sigma(\rho_0) \qquad \text{Equation of state}$$

Linearized Euler's equations

$$\rho_0 D_0 \delta u + \nabla \delta p + \rho_0 (\delta u \cdot \nabla) u_0 - \frac{\nabla p_0}{\rho_0} \delta\rho = \rho_0 \delta\varphi$$

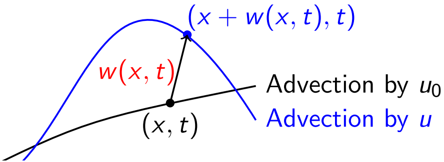
$$\rho_0 D_0 \left(\frac{\delta\rho}{\rho_0} \right) + \nabla \cdot (\rho_0 \delta u) = 0$$

$$\delta p = c_0^2 \delta\rho$$

- Standard model for sound propagation in moving fluids

Derivation of Galbrun's equation—Lagrangian displacement w

$$(\partial_t + \mathcal{L}_{u_0})w \equiv D_0 w - (w \cdot \nabla)u_0 = \delta u$$



- $\mathcal{L}_{u_0} w = (u_0 \cdot \nabla)w - (w \cdot \nabla)u_0$, Lie derivative on vector
- On *scalars* $D_0 = \partial_t + u_0 \cdot \nabla = \partial_t + \mathcal{L}_{u_0}$
- $\nabla \cdot (\rho_0 \delta u) = \rho_0 D_0(\rho_0^{-1} \nabla \cdot (\rho_0 w)) \implies$

$$0 = \rho_0 D_0 \left(\frac{\delta\rho}{\rho_0}\right) + \nabla \cdot (\rho_0 \delta u) = \rho_0 D_0 \left(\frac{\delta\rho + \nabla \cdot (\rho_0 w)}{\rho_0}\right)$$

- **"No resonance" assumption:** $\delta\rho = -\nabla \cdot (\rho_0 w)$

Galbrun's equation (Galbrun 1931)

$$\rho_0 D_0 \delta u_w + \nabla(\rho_0 c_0 \delta\hat{\rho}_w) + \rho_0(\delta u_w \cdot \nabla) u_0 - c_0(\nabla \rho_0)\delta\hat{\rho}_w = \rho_0 \delta\varphi$$

- $\delta u_w = D_0 w - (w \cdot \nabla) u_0$
- $\delta\hat{\rho}_w = \rho_0^{-1} c_0 \delta\rho_w = -\rho_0^{-1} c_0 \nabla \cdot (\rho_0 w)$
- Vector wave equation
  - $D_0^2 w - c_0^2 \nabla(\nabla \cdot w) = \delta\varphi$ (Homogeneous background flow)
- Perturbations of Newtonian stars (e.g. Friedman & Schutz 1978)
- Wave energy conservation law (e.g. Friedman & Schutz 1978)
- Derivable from variational principle (e.g. Friedman & Schutz 1978)
- Well-posedness in 2D of regularized formulations
  - General time-dependence, homogeneous background flow (Berriri et al. 2006)
  - Harmonic time-dependence, general time independent background flow (Bonnet-Ben Dhia et al. 2012)

Informal sketch of our approach

- Solve linearized Euler's equations for δu and $\delta\rho$
- Solve equation for w
- If the "no resonance" assumption is fulfilled ($-\nabla \cdot (\rho_0 w) = \delta\rho$), then w solves Galbrun's equation

The "no resonance" assumption scrutinized

- Introduce $h = \rho_0^{-1}(\delta\rho + \nabla \cdot (\rho_0 w))$
- "No resonance" assumption: $\rho_0 D_0 h = 0 \implies h \equiv 0$

$$0 = 2 \int_{\Omega} \rho_0 h \, D_0 h = \frac{d}{dt} \int_{\Omega} \rho_0 h^2 + \int_{\partial\Omega} \rho_0 (n \cdot u_0) h^2$$

- Partition $\partial\Omega = \Gamma_- \cup \Gamma_0 \cup \Gamma_+$ based on the sign of $n \cdot u_0$

$$\frac{d}{dt} \int_{\Omega} \rho_0 h^2 = \int_{\Gamma_-} \rho_0 |n \cdot u_0| h^2 - \int_{\Gamma_+} \rho_0 |n \cdot u_0| h^2 \leq \int_{\Gamma_-} \rho_0 |n \cdot u_0| h^2$$

Thus, $h \equiv 0$ iff $h|_{t=0} = 0$ and $h|_{\Gamma_-} = 0$

“No resonance” assumption—Imposing vanishing data for h

- Assume δu and $\delta\rho$ solve linearized Euler's equations
- Let $\tilde{w}(t)$ satisfy $\nabla \cdot (\rho_0 \tilde{w}(t)) = -\delta\rho(t)$
- Impose $w|_{t=0} = \tilde{w}(0)$, then

$$h|_{t=0} = (\delta\rho + \nabla \cdot (\rho_0 w))|_{t=0} = \delta\rho(0) + \nabla \cdot (\rho_0 \tilde{w}(0)) = 0$$

- Impose $w|_{\Gamma_-} = \tilde{w}|_{\Gamma_-}$, then ?
- Impose $\nabla \cdot (\rho_0 w)|_{\Gamma_-} = \nabla \cdot (\rho_0 \tilde{w})|_{\Gamma_-}$, then ?
- Resort to $n \cdot u_0 = 0$ on all of $\partial\Omega$

Well-posedness of linearized Euler's equations

- $\Omega \subset \mathbb{R}^d$ open bounded domain with C^1 boundary
- time independent background flow
- $n \cdot u_0 = 0$ on $\partial\Omega = \Gamma_0$
- background flow quantities and their first order derivatives essentially bounded
- $\operatorname{ess\,inf} \rho_0 > 0$ and $\operatorname{ess\,inf} c_0 > 0$
- $Y : \partial\Omega \to [0, \infty)$ Lipschitz continuous admittance function

$$\left(\rho_0 \partial_t + \begin{pmatrix} \rho_0 u_0 \cdot \nabla & \nabla(\rho_0 c_0 \cdot) \\ \rho_0 c_0 \nabla \cdot & \rho_0 u_0 \cdot \nabla \end{pmatrix} + \rho_0 c_0 \begin{pmatrix} \frac{\nabla u_0}{c_0} & -\frac{\nabla \rho_0}{\rho_0} \\ \frac{\nabla \rho_0}{\rho_0} \cdot & -\frac{u_0 \cdot \nabla c_0}{c_0^2} \end{pmatrix}\right) \begin{pmatrix} \delta u \\ \delta \hat{\rho} \end{pmatrix} = \rho_0 \begin{pmatrix} \delta \varphi \\ 0 \end{pmatrix}$$

Initial condition $$\begin{pmatrix} \delta u \\ \delta \rho \end{pmatrix} = \begin{pmatrix} \delta u_0 \\ \delta \rho_0 \end{pmatrix}$$

Boundary condition $$Y \delta \hat{\rho} - n \cdot \delta u = g$$

Theoretical framework—Friedrichs' systems

(Ern et al. 2007 and Burazin & Erceg 2016)

- $L, (\cdot,\cdot)_L$ real Hilbert space identified with its dual L'
- $\mathcal{D}$ dense subspace of L
- C generic positive constant
- $T, \tilde{T} : \mathcal{D} \to L$ linear, time independent

$$(T\phi, \psi)_L = (\phi, \tilde{T}\psi)_L \ \forall \phi, \psi \in \mathcal{D} \quad \text{(T1)}$$

$$\|(T + \tilde{T})\phi\|_L \le C\|\phi\|_L \ \forall \phi \in \mathcal{D} \quad \text{(T2)}$$

- W_0 completion of $\mathcal{D}$ in $(\cdot,\cdot)_W = (\cdot,\cdot)_L + (T\cdot, T\cdot)_L$
- Extend by density and adjoints to $T, \tilde{T} \in \mathcal{L}(L; W_0')$
  (detailed in Antonić and Burazin 2010)
- Graph space $W = \{\xi \in L \mid T\xi \in L\}$
- Boundary operator $D \in \mathcal{L}(W; W')$

$$\langle D\xi, \tilde{\xi}\rangle_W = (T\xi, \tilde{\xi})_L - (\xi, \tilde{T}\tilde{\xi})_L = \langle D\tilde{\xi}, \xi\rangle_W$$

- $V, \tilde{V} \subset W$ subspaces

$$\langle D\xi, \xi\rangle_W \ge 0 \text{ for all } \xi \in V \text{ and } \langle D\tilde{\xi}, \tilde{\xi}\rangle_W \le 0 \text{ for all } \tilde{\xi} \in \tilde{V} \quad \text{(V1)}$$

$$V = D(\tilde{V})^0 \text{ and } \tilde{V} = D(V)^0 \quad \text{(V2)}$$

Theoretical framework—Abstract Cauchy problem

(Burazin and Erceg 2016)

$$(\partial_t + T)\xi = f \text{ for } t > 0 \tag{4a}$$

$$\xi = \xi_0 \text{ at } t = 0 \tag{4b}$$

Theorem 1

If T and $\tilde{T}$ satisfy conditions (T1) and (T2); $V \subset W$ and $\tilde{V} \subset W$ satisfy conditions (V1) and (V2); and $f \in L^1((0,\tau);L)$. Then, for every $\xi_0 \in L$ problem (4) has the unique mild solution $\xi \in C([0,\tau];L)$ given by

$$\xi(t) = e^{\lambda_0 t} S_{\lambda_0}(t)\xi_0 + \int_0^t e^{\lambda_0(t-s)} S_{\lambda_0}(t-s) f(s)\,\mathrm{d}s.$$

- $(S_{\lambda_0}(t))_{t\geq 0}$ contraction C_0-semigroup generated by $-(T+\lambda_0 I)|_V : V \subset L \to L$
- $\lambda_0 = \max\left\{0, -\inf_{\phi\in\mathcal{D}\setminus\{0\}} \frac{((T+\tilde{T})\phi,\phi)_L}{\|\phi\|_L^2}\right\} \leq \sup_{\phi\in\mathcal{D}\setminus\{0\}} \frac{\|(T+\tilde{T})\phi\|_L}{\|\phi\|_L} < \infty$
- $\|\xi(t)\|_L \leq e^{\lambda_0\tau}\left(\|\xi_0\|_L + \int_0^t \|f(s)\|_L\,\mathrm{d}s\right)$

Linearized Euler's equations

- $L = L^2_{\rho_0}(\Omega)^{d+1} = L^2(\Omega)^{d+1}$, $(\cdot,\cdot)_L = (\rho_0 \cdot, \cdot)$
- $\mathcal{D} = \mathring{C}^\infty(\Omega)^{d+1}$
- $\rho_0 T = A + B = \begin{pmatrix} \rho_0 u_0 \cdot \nabla & \nabla(\rho_0 c_0 \cdot) \\ \rho_0 c_0 \nabla \cdot & \rho_0 u_0 \cdot \nabla \end{pmatrix} + \rho_0 c_0 \begin{pmatrix} \dfrac{\nabla u_0}{c_0} & -\dfrac{\nabla \rho_0}{\rho_0} \\ \dfrac{\nabla \rho_0}{\rho_0} \cdot & -\dfrac{u_0 \cdot \nabla c_0}{c_0^2} \end{pmatrix}$
- $\rho_0 \tilde{T} = -A + B^T$
- Conditions (T1) and (T2) satisfied
- $W = \{\xi \in L \mid A\xi \in L\}$
- $\langle D\phi, \psi \rangle_W = (A\phi, \psi) + (\phi, A\psi) = \int_{\partial\Omega} \psi \cdot A(n)\phi; \quad \phi, \psi \in C^\infty(\bar{\Omega})^{d+1}$
  - $A(n) = \begin{pmatrix} 0 & n \\ n\cdot & 0 \end{pmatrix}$

Linearized Euler's equations—Boundary conditions

Following Rauch (1985)

- For $x \in \partial\Omega$ define
  $N(x) = \{\xi \in \mathbb{R}^{d+1} \mid -n(x)\cdot\xi_1 + Y(x)\xi_2 = 0\}$
  $\tilde{N}(x) = \{\xi \in \mathbb{R}^{d+1} \mid \quad n(x)\cdot\xi_1 + Y(x)\xi_2 = 0\}$
- $\tilde{N} = (A(n)N)^\perp$ and $N = (A(n)\tilde{N})^\perp$
- $\ker A(n) = \{\xi \mid n\cdot\xi_1 = 0 \text{ and } \xi_2 = 0\} \subset N \cap \tilde{N}$,
  $\dim \ker A(n) = d - 1$
- Assume $N(x) = \operatorname{span}(v_i(x))_{i=1}^d$, $v_i : \partial\Omega \to \mathbb{R}^{d+1}$ Lipschitz
- Incorporate inhomogeneous BCs in f

Linearized Euler's equations—Boundary conditions

Following Rauch (1985)

- $\gamma_N : W \to H^{-1/2}(\partial\Omega; \mathbb{R}^{d+1}/N(x))$
  - $\phi \in C^1(\overline{\Omega})$; $\gamma_N \phi = 0$ iff $\phi(x) \in N(x)\ \forall x \in \partial\Omega$
- $V = \{\xi \in W \mid \gamma_N \xi = 0\}$
- $\gamma_{\tilde{N}} : W \to H^{-1/2}(\partial\Omega; \mathbb{R}^{d+1}/\tilde{N}(x))$
  - $\phi \in C^1(\overline{\Omega})$; $\gamma_{\tilde{N}} \phi = 0$ iff $\phi(x) \in \tilde{N}(x)\ \forall x \in \partial\Omega$
- $\tilde{V} = \{\xi \in W \mid \gamma_{\tilde{N}} \xi = 0\}$

Theorem 2

$V \cap C^1(\overline{\Omega})$ is dense in V
$\tilde{V} \cap C^1(\overline{\Omega})$ is dense in $\tilde{V}$

Proof.

Rauch's Theorem 4 □

Linearized Euler's equations—Boundary conditions

Theorem 3

For all pairs $\xi \in V$ *and* $\tilde{\xi} \in \tilde{V}$ *it holds that* $\langle D\xi, \tilde{\xi} \rangle_W = 0$

Proof.

By theorem 2

$$\langle D\xi, \tilde{\xi} \rangle_W = (A\xi, \tilde{\xi}) + (\xi, A\tilde{\xi}) = \lim_{k\to\infty} \lim_{l\to\infty} (A\xi_k, \tilde{\xi}_l) + (\xi_k, A\tilde{\xi}_l) =$$

$$\lim_{k\to\infty} \lim_{l\to\infty} \underbrace{(A(n)\xi_k, \tilde{\xi}_l)_{\partial\Omega}}_{=0} = 0$$

□

Theorem 4

If $\xi \in W$ *and* $\langle D\xi, \tilde{\xi} \rangle_W = 0$ *for all* $\tilde{\xi} \in \tilde{V}$, *then* $\xi \in V$

If $\tilde{\xi} \in W$ *and* $\langle D\tilde{\xi}, \xi \rangle_W = 0$ *for all* $\xi \in V$, *then* $\tilde{\xi} \in \tilde{V}$

Proof.

$\tilde{V} \cap Lip(\overline{\Omega}) \subset \tilde{V}$, thus Theorem 3 yields

$0 = \langle D\xi, \tilde{\xi} \rangle = (A\xi, \tilde{\xi}) + (\xi, A\tilde{\xi})$ for all $\tilde{\xi} \in \tilde{V} \cap Lip(\overline{\Omega})$

Result follows by Rauch's Proposition 3

□

Well-posedness of Linearized Euler's equations

Theorem 2 yields (V1)

- $\langle D\xi, \xi \rangle_W = \lim_{k\to\infty} (A\xi_k, \xi_k) + (\xi_k, A\xi_k) = \lim_{k\to\infty} (A(n)\xi_k, \xi_k)_{\partial\Omega} = \lim_{k\to\infty} 2(Y\xi_{2,k}, \xi_{2,k})_{\partial\Omega} \geq 0 \ \forall \xi \in V$
- $\langle D\tilde{\xi}, \tilde{\xi} \rangle_W = \lim_{k\to\infty} (A\tilde{\xi}_k, \tilde{\xi}_k) + (\tilde{\xi}_k, A\tilde{\xi}_k) = \lim_{k\to\infty} (A(n)\tilde{\xi}_k, \tilde{\xi}_k)_{\partial\Omega} = \lim_{k\to\infty} -2(Y\tilde{\xi}_{2,k}, \tilde{\xi}_{2,k})_{\partial\Omega} \leq 0 \ \forall \tilde{\xi} \in \tilde{V}$

Note that $D(\tilde{V})^0 = \{\xi \in W'' = W \mid 0 = \langle D\tilde{\xi}, \xi \rangle_W \, \forall \tilde{\xi} \in \tilde{V}\}$ and $D(V)^0 = \{\tilde{\xi} \in W \mid 0 = \langle D\xi, \tilde{\xi} \rangle_W \, \forall \xi \in V\}$, thus Theorems 3 and 4 yield (V2)

Thus, Theorem 1 applies

Lagrangian displacement is well-defined

- $n \cdot u_0 = 0$ on $\partial\Omega$
- $\delta u \in L^1((0,\tau); L^2(\Omega)^d)$

$$\begin{aligned} \partial_t + (u_0 \cdot \nabla)w - (w \cdot \nabla)u_0 = \delta u \text{ in } \Omega \text{ for all } t > 0 \\ w = w_0 \text{ in } \Omega \text{ at } t = 0 \end{aligned}$$

- $L = L_{\rho_0}(\Omega)^d$
- $\mathcal{D} = \mathring{C}^\infty(\Omega)^d$
- $T = u_0 \cdot \nabla - \nabla u_0,\ \tilde{T} = -u_0 \cdot \nabla - (\nabla u_0)^T$
  - Conditions (T1) and (T2) hold
- $V = \tilde{V} = W$
  - Conditions (V1) and (V2) hold

Thus, Theorem 1 applies. If w is sufficiently regular and $\nabla \cdot (\rho_0 w_0) = -\delta\rho_0$, then w solves Galbruns equation

Galbrun's equation—Energy estimate

First order form of Galbrun's equation

$$\rho_0 D_0 \delta u + \nabla(\rho_0 c_0\, \delta\hat{\rho}) + \rho_0(\delta u \cdot \nabla) u_0 - c_0(\nabla \rho_0)\delta\hat{\rho} = \rho_0 \delta\varphi \quad (6a)$$

$$D_0 w - (w \cdot \nabla) u_0 - \delta u = 0 \quad (6b)$$

$$c_0 \rho_0^{-1} \nabla \cdot (\rho_0 w) + \delta\hat{\rho} = 0 \quad (6c)$$

- Integrate by parts in $(\delta u, (6a)) + \tau_0^{-2}(\rho_0 w, (6b))$
  - $\tau_0 \in (0, \infty)$ time scale
- 2nd term (6a)

$$(\delta u, \nabla(\rho_0 c_0 \delta\hat{\rho})) = -(\nabla \cdot \delta u, \rho_0 c_0 \delta\hat{\rho}) + (n \cdot \delta u, \rho_0 c_0 \delta\hat{\rho})_{\partial\Omega}$$

$$\nabla \cdot (\rho_0 \delta u) = \rho_0 D_0 (\underbrace{\rho_0^{-1} \nabla \cdot (\rho_0 w)}_{= -c_0^{-1}\delta\hat{\rho}}) \quad (\nabla \cdot (6b))$$

  - Employ formal skew symmetry of $\rho_0 u_0 \cdot \nabla$

Galbrun's equation—Energy estimate

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\tau_0^{-2}(\rho_0 w, w) + (\rho_0\delta\hat{\rho}, \delta\hat{\rho}) + (\rho_0\delta u, \delta u)\right) \\
&= \tau_0^{-2}(\rho_0 w, (\nabla u_0)w) + \left(\rho_0\delta\hat{\rho}, \frac{u_0\cdot\nabla c_0}{c_0}\delta\hat{\rho}\right) - (\rho_0\delta u, (\nabla u_0)\delta u) \\
&+ \tau_0^{-2}(\rho_0\delta u, w) - \frac{1}{2}\int_{\partial\Omega}\rho_0\xi\cdot A(n)\xi + (\rho_0\delta u, \delta\varphi)
\end{aligned}
$$

where

$$
\xi\cdot A(n)\xi = \begin{pmatrix}\delta u \\ \delta\hat{\rho} \\ \tau_0^{-1}w\end{pmatrix}\cdot\begin{pmatrix} n\cdot u_0 & c_0 n & 0 \\ c_0 n\cdot & n\cdot u_0 & 0 \\ 0 & 0 & n\cdot u_0\end{pmatrix}\begin{pmatrix}\delta u \\ \delta\hat{\rho} \\ \tau_0^{-1}w\end{pmatrix}
$$

- Same as for linearized Euler's equations (3) appended with definition (4) of w (however, then $\delta\hat{\rho} \neq c_0\rho_0^{-1}\nabla\cdot(\rho_0 w)$)

Galbrun's equation—Energy estimate

- $n \cdot u_0 = 0$ on $\partial\Omega$
- $w = w_0$ and $\partial_t w = \dot{w}_0$ in Ω at $t = 0$
- $-n \cdot (D_0 w - (w \cdot \nabla) u_0) - Y c_0 \rho_0^{-1} \nabla \cdot (\rho_0 w) = g$ on $\partial\Omega$ for all $t > 0$
  - $-n \cdot \delta u + Y \delta\hat{\rho} = g$
  - If $g(x) \neq 0$, then $Y(X) \geq a > 0$

$$
\tau_0^{-2} \|w(t)\|_{\rho_0}^2 + \|\delta\hat{\rho}(t)\|_{\rho_0}^2 + \|\delta u(t)\|_{\rho_0}^2 \leq
$$

$$
C e^{\gamma t} \left(\tau_0^{-2} \|w_0\|_{\rho_0}^2 + \|\delta\hat{\rho}_0\|_{\rho_0}^2 + \|\delta u_0\|_{\rho_0}^2 + \int_0^t \|\delta\varphi(s)\|_{\rho_0}^2 + \|g(s)\|_{\partial\Omega,\rho_0}^2 \, \mathrm{d}s \right)
$$

$$
\tau_0^{-2} \|w(t)\|_{\rho_0}^2 + \|c_0 \rho_0^{-1} \nabla \cdot (\rho_0 w)(t)\|_{\rho_0}^2 + \|(\partial_t + \mathcal{L}_{u_0}) w(t)\|_{\rho_0}^2 \leq
$$

$$
C e^{\gamma t} \left(\tau_0^{-2} \|w_0\|_{\rho_0}^2 + \|c_0 \rho_0^{-1} \nabla \cdot (\rho_0 w_0)\|_{\rho_0}^2 + \|\dot{w}_0 + \mathcal{L}_{u_0} w_0\|_{\rho_0}^2 + \int_0^t \|\delta\varphi(s)\|_{\rho_0}^2 + \|g(s)\|_{\partial\Omega,\rho_0}^2 \, \mathrm{d}s \right)
$$

Conclusions

- $n \cdot u_0 = 0$ on $\partial\Omega$
- Well-posedness of linearized Euler's equations
  - Mild solutions
  - Energy estimate (strong solutions)
- Well-posedness of Galbrun's equation
  - Existence from linearized Euler's equations (if w regular enough)
  - Energy estimate (strong solutions)