Adaptive Space-Time Isogeometric Analysis of Parabolic Evolution Problems

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Joint work with Ulrich Langer[1,2] and Sergey Repin[3]

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Outline

- Introduction
  1. model problem
  2. a posteriori error estimates and the identity
- Globally stabilised space-time IgA schemes [Langer, Moore, and Neumüller, 2016]
- **Locally stabilized** space-time IgA schemes
- Adaptive space-time IgA schemes
- Numerical results
- Conclusions and roadmap

Intro: model problem and a posteriori error estimates and identity

Parabolic I-BVP model problem

Find $u : \overline{Q} \to \mathbb{R}$ satisfying the *linear parabolic initial-boundary value problem (I-BVP)*

$$\begin{aligned} \partial_t u - \mathrm{div}_x \nabla_x u &= f && \text{in } Q, \\ u(x, 0) &= u_0 && \text{on } \Sigma_0, \\ u = u_D &= 0 && \text{on } \Sigma, \end{aligned}$$

$\Omega \subset \mathbb{R}^d$, $d = \{1, 2, 3\}$, $T > 0$
$Q := \Omega \times (0, T)$
$\partial Q := \Sigma \cup \overline{\Sigma}_0 \cup \overline{\Sigma}_T$
$\Sigma := \partial\Omega \times (0, T)$
$\Sigma_0 := \Omega \times \{0\}$
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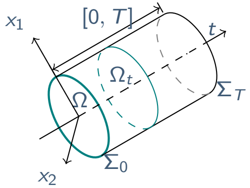

where ∂_t is the time derivative,
$\Delta_x = \mathrm{div}_x \nabla_x$, div_x and ∇_x are
Laplace, divergence, and gradient operators in space, resp.,
$u_0 \in H^1_0(\Sigma_0)$ is a given initial state,
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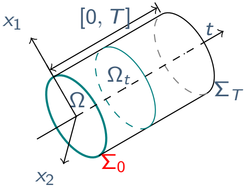

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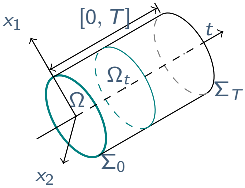

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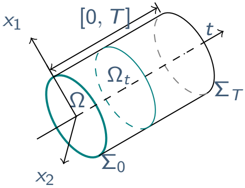

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Solvability results [Ladyzhenskaya, 1954]

- **Weak formulation**:

  Find $u \in H_0^{1,0}(Q) := \{ v \in L^2(Q) \,:\, \nabla_x v \in [L^2(Q)]^d,\ v|_{\Sigma} = 0 \}$ satisfying

  $$(\star) \quad a(u, w) = \ell(w), \quad \forall w \in H_{0,\overline{0}}^{1,1}(Q) := \{ v \in H_0^{1,0}(Q) \,:\, \partial_t v \in L^2(Q),\ v|_{\overline{\Sigma}_T} = 0 \},$$

  where

  $$a(u, w) := \big(\nabla_x u, \nabla_x w\big)_Q - \big(u, \partial_t w\big)_Q,$$
  $$\ell(w) := \big(f, w\big)_Q + (u_0, w)_{\Sigma_0}.$$

- If $f \in L^{2,1}(Q_T) := \left\{ v \in L^1(Q) : \int_0^T \|v(\cdot, t)\|_{L^2(\Omega)}\, \mathrm{d}t < \infty \right\}$ and $u_0 \in L^2(\Omega)$, then there $\exists$ a unique weak solution $u \in H_0^{1,0}(Q)$ of $(\star)$ that also belongs to $V_0^{1,0} := C([0, T]; L^2(\Omega)) \cap H_0^{1,0}(Q)$.

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Functional a posteriori error analysis **[Repin, 2002]**

- For any $v \in H_0^{1,1}(Q)$, $\boldsymbol{y} \in H^{\mathrm{div}_x,0}(Q) := \{\boldsymbol{y} \in [L^2(Q)]^{d+1} : \mathrm{div}_x \boldsymbol{y} \in L^2(Q)\}$, and $\beta > 0$, we have the following **functional a posteriori error estimate**:

$$||| u - v |||^2 := \|\nabla_x (u - v)\|^2_Q + \| u - v \|^2_{\Sigma_T} \leq \overline{\mathrm{M}}^{\mathrm{I},2}(v, \boldsymbol{y}; \beta)$$

with the majorant

$$\overline{\mathrm{M}}^{\mathrm{I},2}(v, \boldsymbol{y}; \beta) := (1+\beta) \underbrace{\|\boldsymbol{y} - \nabla_x v\|^2_Q}_{\text{dual term } \overline{\mathrm{m}}^{\mathrm{I}}_{\mathrm{d}}} + (1+\tfrac{1}{\beta})\, C^2_{\mathrm{F}\Omega} \underbrace{\|f + \mathrm{div}_x \boldsymbol{y} - \partial_t v\|^2_Q}_{\text{equilibration/reliability term } \overline{\mathrm{m}}^{\mathrm{I}}_{\mathrm{eq}}} .$$

- Main properties:
  - **universal** for any v from admissible functional space,
  - **computable**,
  - **reliable** and **realistic** w.r.t. the error, i.e., $1 \leq I_{\mathrm{eff}} = \frac{\overline{\mathrm{M}}}{|||u-v|||}$ is close to 1,
  - **efficient for adaptive strategies** $V_h \to V_{h_{\mathrm{ref}}}$,
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Stronger solvability results **[Ladyzhenskaya, 1954]**

- If $f \in L^2(Q)$ and $u_0 \in H^1_0(\Omega)$,
  then the I-BVP is **uniquely solvable** in

$$H_0^{\Delta_x,1}(Q) := \Big\{ u \in H_0^{1,1}(Q) \,:\, \Delta_x u \in L^2(Q) \Big\},$$

  and u continuously depends on t in the $H^1_0(\Omega)$-norm.

- **Maximal parabolic regularity** for $\partial_t u - \mathrm{div}_x(A(x,t)\nabla_x u) = f$:
  for $f \in X = L^p((0,T); L^q(\Omega))$, $1 < p, q < \infty$ and $u_0 = 0$
  there $\exists$ $C > 0$, such that

$$\|\partial_t u\|_X + \|\mathrm{div}_x(A(x,t)\nabla_x u)\|_X \leq C\,\|f\|_X.$$

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Error identity [Anjam and Pauly, 2016]

For any $v \in H_0^{\Delta_x,1}(Q)$ approximating $u \in H_0^{\Delta_x,1}(Q)$, we have the **error identity**:

$$\|\Delta_x(u-v)\|_Q^2 + \|\partial_t(u-v)\|_Q^2 + \|\nabla_x(u-v)\|_{\Sigma_T}^2$$

$$=: \|\!|u-v|\!\|_{\mathcal{L},Q}^2 \equiv \mathbb{E}\mathrm{d}^2(v)$$

$$:= \|\nabla_x(u_0-v)\|_{\Sigma_0}^2 + \|\Delta_x v + f - \partial_t v\|_Q^2.$$

Note:

⊕ reconstruction of $\mathbb{E}\mathrm{d}^2(v)$ **does not include time overhead**

⊖ **extra regularity** $u, v \in H_0^{\Delta_x,1}(Q)$ is required (not practival for FEM) ⇒ but natural for IgA framework!

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Globally stabilized space-time IgA schemes

IgA framework [Hughes et al., 2005]

Physical domain $Q \subset \mathbb{R}^{d+1}$ (single patch) is defined from

Parametric domain $\widehat{Q} := (0,1)^{d+1}$ by the

Geometrical mapping $\Phi : \widehat{Q} \to Q = \Phi(\widehat{Q}) \subset \mathbb{R}^{d+1}, \quad \Phi(\xi) = \sum_{i \in \mathcal{I}} \widehat{B}_{i,p}(\xi)\, \mathbf{P}_i,$

- $\widehat{B}_{i,p}, i \in \mathcal{I}$, are the B-splines, NURBS, THB-splines;
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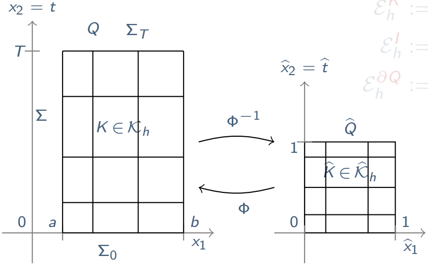

Set of facets:

$$\mathcal{E}_h^K := \{E \in \mathcal{E}_h : E \cap \partial K \neq \emptyset, K \in \mathcal{K}_h\},$$

$$\mathcal{E}_h^I := \{E : \exists\, K, K' \in \mathcal{K}_h : E = \partial K \cap \partial K', E \not\subset \partial Q\},$$

$$\mathcal{E}_h^{\partial Q} := \{E : \exists\, K, K' \in \mathcal{K}_h : E = \partial K \cap \partial K', E \cap \partial Q \neq \emptyset\},$$

where $\mathcal{E}_h^{\partial Q} := \mathcal{E}_h^{\Sigma} \cup \mathcal{E}_h^{\Sigma_T} \cup \mathcal{E}_h^{\Sigma_0}$, such that

$$\mathcal{E}_h^{\Sigma_*} := \{E : \exists\, K, K' \in \mathcal{K}_h : E = \partial K \cap \partial K', E \cap \Sigma_* \neq \emptyset\}.$$

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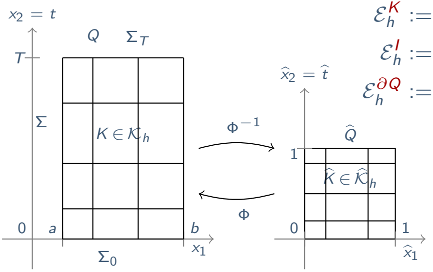

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Fundamentals [Bazilevs et. all., 2006],[Evans and Hughes, 2013]

Let $K \in \mathcal{K}_h$ and $h_K := \mathrm{diam}_{K \in \mathcal{K}_h}(K)$, then the ***inverse inequalities***

$$\|v_h\|_{\partial K} \leq C_{int,0}\, h_K^{-1/2}\, \|v_h\|_K \quad \text{and} \quad \|\nabla v_h\|_K \leq C_{int,1}\, h_K^{-1}\, \|v_h\|_K$$

hold for all $v_h \in V_h := \mathrm{span}\left\{\phi_{h,i} := \widehat{B}_{i,p} \circ \Phi^{-1}\right\}_{i \in \mathcal{I}}$, where $C_{int,0}, C_{int,1} > 0$ are constants independent of K.

Let $K \in \mathcal{K}_h$, then the ***scaled trace inequality***

$$\|v\|_{\partial K} \leq C_{tr}\, h_K^{-1/2}(\|v\|_K + h_K\, \|\nabla v\|_K)$$

hold for all $v \in H^1(K)$, where $C_{tr} > 0$ is a constant independent of K.

Approximation error estimates

Let $\ell, s \in \mathbb{N}$ be $0 \leq \ell \leq s \leq p+1$, $u \in H^s_{0,\underline{0}}(Q)$, and K and $\underline{K}$ are element and its extension, resp. Then, $\exists\, \Pi_h : H^s_{0,\underline{0}}(Q) \to V_{0h}$ such that

$$|v - \Pi_h v|^2_{H^\ell(K)} \leq C^2_{\ell,s} h_K^{2(s-\ell)} \sum_{i=0}^{s} c_K^{2(i-\ell)}\, |v|^2_{H^i(\underline{K})}, \quad \forall v \in L^2(Q) \cap H^\ell(\underline{K}),$$

where $c_K := \|\nabla_x \Phi\|_{L^\infty(\Phi^{-1}(\underline{\hat{K}}))}$, and $C_{\ell,s} > 0$ is a constant dependent on s, ℓ, p, and the shape regularity of K, described by Φ and $\nabla_x \Phi$.

Stabilized variational identity for parabolic I-BVP

Testing the $-\Delta_x u + \partial_t u = f$ ($(d+1)$-dimetional elliptic problem with convection in $(d+1)$th direction) by the upwind test function with $\lambda, \mu \geq 0$

$$\lambda\, w + \mu\, \partial_t w, \quad w \in H^{\nabla_x \partial_t, 1}_{0,\underline{0}}(Q) := \left\{ w \in H^{\Delta_x, 1}_{0,\underline{0}}(Q) : \nabla_x \partial_t w \in L^2(Q) \right\},$$

we obtain the variational identity

$$a(u, \lambda\, w + \mu\, \partial_t w) =: \boldsymbol{a_s(u, w) = \ell_s(w)} := \ell(\lambda\, w + \mu\, \partial_t w)_Q, \quad \forall w \in H^{\nabla_x \partial_t, 1}_{0,\underline{0}}(Q)$$

for the solution $u \in H^{\Delta_x, 1}_{0,\underline{0}}(Q)$.

For any $v \in H^{\Delta_x, 1}_{0,\underline{0}}$ approximating u, the error $u - v$ is measured in terms of the norm

$$\| u - v \|_s^2 := \lambda \left(\|\nabla_x (u - v)\|_Q^2 + \| u - v \|_{\Sigma_T}^2 \right) + \mu \left(\|\partial_t (u - v)\|_Q^2 + \| \nabla_x (u - v) \|_{\Sigma_T}^2 \right).$$

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$$\| u - v \|^2_s := \lambda \left(\|\nabla_x (u - v)\|^2_Q + \| u - v \|^2_{\Sigma_T} \right) + \mu \left(\|\partial_t (u - v)\|^2_Q + \| \nabla_x (u - v) \|^2_{\Sigma_T} \right).$$

Stabilized variational identity for parabolic I-BVP

Testing the $-\Delta_x u + \partial_t u = f$ ($(d+1)$-dimetional elliptic problem with convection in $(d+1)$th direction) by the upwind test function with $\lambda = 1$ and $\mu = \delta_h = \theta h,\ \theta > 0,\ h := \max_{K \in \mathcal{K}_h} \{ h_K \}$

$$w + \delta_h \partial_t w, \quad w \in H_{0,\underline{0}}^{\nabla_x \partial_t, 1}(Q) := \{ w \in H_{0,\underline{0}}^{\Delta_x, 1}(Q) : \nabla_x \partial_t w \in L^2(Q) \},$$

we obtain the variational identity

$$a(u, w + \delta_h \partial_t w) =: \boldsymbol{a_{s,h}(u, w) = \ell_{s,h}(w)} := \ell(w + \delta_h \partial_t w)_Q, \quad \forall w \in H_{0,\underline{0}}^{\nabla_x \partial_t, 1}(Q)$$

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$$\| u - v \|_{s,h}^2 := \| \nabla_x (u - v) \|_Q^2 + \| u - v \|_{\Sigma_T}^2 + \delta_h \Big(\| \partial_t (u - v) \|_Q^2 + \| \nabla_x (u - v) \|_{\Sigma_T}^2 \Big).$$

Stabilized space-time IgA scheme

IgA scheme [Langer, Moore, and Neumüller, 2016]

Find $u_h \in V_{0h} := \operatorname{span}\left\{\phi_{h,i} := \widehat{B}_{i,p} \circ \Phi^{-1}\right\}_{i \in \mathcal{I}} \cap H^1_0(Q) \subset H^{\Delta_x,1}_{0,\underline{0}}(Q)$, i.e. $p \geq 2$, satisfying

$$a_{s,h}(u_h, v_h) = \ell_{s,h}(v_h), \quad \forall v_h \in V_{0h},$$

where

$$a_{s,h}(u_h, v_h) := \left(\partial_t u_h, v_h + \delta_h \, \partial_t v_h\right)_Q + \left(\nabla_x u_h, \nabla_x (v_h + \delta_h \, \partial_t v_h)\right)_Q,$$

$$\ell_{s,h}(v_h) := (f, v_h + \delta_h \, v_h)_Q,$$

Remark: For $u \in V^s_0, s \geq 2$ and $u_h \in V_{0h}$,
there exists **a priori error estimates of the form**

$$\|u - u_h\|_{s,h} \leq C \, h^{r-1} \, \|u\|_{H^r(Q)}, \quad C > 0 \quad \text{and} \quad r = \min\{s, p+1\}.$$

Majorant of the error norm $|\!|\!| \cdot |\!|\!|_s$

Theorem 1 [Langer, Matculevich, and Repin, 2016]

For any approximation $v \in H_0^{\Delta_x,1}(Q)$ to $u \in H_0^{\Delta_x,1}(Q)$ and for any $\mathbf{y} \in H^{\mathrm{div}_x,0}(Q)$, the error $e = u - v$ can be estimated as follows:

$$\lambda \left(\|\nabla_x e\|_Q^2 + \|e\|_{\Sigma_T}^2 \right) + \mu \left(\|\partial_t e\|_Q^2 + \|\nabla_x e\|_{\Sigma_T}^2 \right) =: |\!|\!| e |\!|\!|_s^2$$

$$\leq \overline{\mathrm{M}}_s^{\mathrm{I},2}(v, \mathbf{y}; \beta, \alpha) := \lambda \, \overline{\mathrm{M}}^{\mathrm{I},2}(v, \mathbf{y}; \beta) + \mu \Big((1 + \alpha) \, \|\mathrm{div}_x \mathbf{r}_\mathrm{d}\|_Q^2 + (1 + \tfrac{1}{\alpha}) \, \|\mathbf{r}_\mathrm{eq}\|_Q^2 \Big)$$

where

$$\begin{aligned} \mathbf{r}_\mathrm{eq}(v, \mathbf{y}) &= f + \mathrm{div}_x \mathbf{y} - \partial_t v & \Leftarrow \quad \partial_t u - \mathrm{div}_x \mathbf{p} &= f, \\ \mathbf{r}_\mathrm{d}(v, \mathbf{y}) &= \mathbf{y} - \nabla_x v & \Leftarrow \quad \mathbf{p} &= \nabla_x u, \\ \mathrm{div}_x \mathbf{r}_\mathrm{d}(v, \mathbf{y}) &= \mathrm{div}_x \mathbf{y} - \Delta_x v. & & \end{aligned}$$

λ, $\mu > 0$, and $\beta, \alpha > 0$ are auxiliary parameters.

Majorant of the error norm $\|\!|\cdot|\!\|_s$

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where

$$\mathbf{r}_{\mathrm{eq}}(v, \mathbf{y}) = f + \mathrm{div}_x \mathbf{y} - \partial_t v \quad \Leftarrow \quad \partial_t u - \mathrm{div}_x \mathbf{p} = f,$$
$$\mathbf{r}_{\mathrm{d}}(v, \mathbf{y}) = \mathbf{y} - \nabla_x v \quad \Leftarrow \quad \mathbf{p} = \nabla_x u,$$
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Majorant of the error norm $\|\!| \cdot |\!\|_{s,h}$

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For any approximation $v \in H_0^{\Delta_x,1}(Q)$ to $u \in H_0^{\Delta_x,1}(Q)$ and for any $\mathbf{y} \in H^{\mathrm{div}_x,0}(Q)$, the error $e = u - v$ can be estimated as follows:

$$\left(\|\nabla_x e\|^2_Q + \|e\|^2_{\Sigma_T}\right) + \delta_h \left(\|\partial_t e\|^2_Q + \|\nabla_x e\|^2_{\Sigma_T}\right) =: \|\!| e |\!\|^2_{s,h}$$

$$\leq \overline{\mathrm{M}}^{\mathrm{I},2}_s(v, \mathbf{y}; \beta, \alpha) := \overline{\mathrm{M}}^{\mathrm{I},2}(v, \mathbf{y}; \beta) + \delta_h \Big((1+\alpha) \, \|\mathrm{div}_x \mathbf{r}_\mathrm{d}\|^2_Q + (1 + \tfrac{1}{\alpha}) \, \|\mathbf{r}_\mathrm{eq}\|^2_Q \Big)$$

where

$$\mathbf{r}_\mathrm{eq}(v, \mathbf{y}) = f + \mathrm{div}_x \mathbf{y} - \partial_t v \qquad \Leftarrow \qquad \partial_t u - \mathrm{div}_x \mathbf{p} = f,$$

$$\mathbf{r}_\mathrm{d}(v, \mathbf{y}) = \mathbf{y} - \nabla_x v \qquad \Leftarrow \qquad \mathbf{p} = \nabla_x u,$$

$$\mathrm{div}_x \mathbf{r}_\mathrm{d}(v, \mathbf{y}) = \mathrm{div}_x \mathbf{y} - \Delta_x v.$$

λ, $\mu > 0$, and $\beta, \alpha > 0$ are auxiliary parameters.

Locally stabilized space-time IgA schemes

Locally stabilized schemes

On each $K \in \mathcal{K}_h$, we test the PDE $\partial_t u - \Delta_x u = f$ with

$$v_h + \delta_K \, \partial_t v_h, \quad \delta_K = \theta_K \, h_K, \quad \text{where} \quad \theta_K > 0 \quad \text{and} \quad h_K := \mathrm{diam}(K),$$

yielding

$$\big(\partial_t u - \Delta_x u, v_h + \delta_K \, \partial_t v_h\big)_K = (f, v_h + \delta_K \, \partial_t v_h)_K, \quad \forall u \in H_0^{\Delta_x,1}(Q), \quad \forall v_h \in V_{0h}.$$

Summing up $K \in \mathcal{K}_h$, we obtain

$$(\partial_t u - \Delta_x u, v_h)_Q + \sum_{K \in \mathcal{K}_h} \delta_K \, (\partial_t u - \Delta_x u, \partial_t v_h)_K =: \boldsymbol{a_{loc}(u, v_h)}$$

$$= \boldsymbol{\ell_{loc}(v_h)} := (f, v_h)_Q + \sum_{K \in \mathcal{K}_h} \delta_K \, (f, \partial_t v_h)_K.$$

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$$\big(\partial_t u - \Delta_x u, v_h\big)_Q + \sum_{K \in \mathcal{K}_h} \delta_K \, \big(\partial_t u - \Delta_x u, \partial_t v_h\big)_K =: \boldsymbol{a_{loc}(u, v_h)}$$

$$= \boldsymbol{\ell_{loc}(v_h)} := \, (f, v_h)_Q + \sum_{K \in \mathcal{K}_h} \delta_K \, (f, \partial_t v_h)_K.$$

Locally Stabilized IgA counterpart

Find $u_h \in V_{0h}$ satisfying the variational IgA scheme

$$a_{loc,h}(u_h, v_h) = \ell_{loc,h}(v_h), \quad \forall u_h, v_h \in V_{0h},$$

where

$$\begin{aligned} a_{loc,h}(u_h, v_h) := (\partial_t u_h, v_h)_Q + (\nabla_x u_h, \nabla_x v_h)_Q \\ + \sum_{K \in \mathcal{K}_h} \delta_K \Big((\partial_t u_h, \partial_t v_h)_K + (\nabla_x u_h, \nabla_x \partial_t v_h)_K \Big) \\ - \sum_{K \in \mathcal{K}_h} \delta_K \sum_{E \in \mathcal{E}_h^K \cap \mathcal{E}_h^I} \big(\boldsymbol{n}_x^E \cdot \nabla_x u_h, \partial_t v_h \big)_E . \end{aligned}$$

and

$$\ell_{loc,h}(v_h) := (f, v_h)_Q + \sum_{K \in \mathcal{K}_h} \delta_K \, (f, \partial_t v_h)_K .$$

V_{0h}-coercivity of $a_{loc,h}(\cdot,\cdot)$

Lemma (coercivity)

Let
$$\theta_K \in \left(0, \frac{h_K}{d\, C_{int,1}^2}\right], \quad K \in \mathcal{K}_h,$$
where $C_{int,1}$ is the constant in the 2nd inverse inequality.
Then, $a_{loc,h}(u_h, v_h) : V_{0h} \times V_{0h} \to \mathbb{R}$ is V_{0h}-coercive w.r.t. to the norm

$$|\!|\!| v_h |\!|\!|_{loc,h}^2 := \|\nabla_x v_h\|_Q^2 + \tfrac{1}{2}\|v_h\|_{\Sigma_T}^2 + \sum_{K\in\mathcal{K}_h} \delta_K\, \|\partial_t v_h\|_K^2,$$

i.e., there exists a constant $\mu_{c,loc} > 0$ independent on K such that
$$a_{loc,h}(u_h, v_h) \geq \mu_{c,loc}\, |\!|\!| v_h |\!|\!|_{loc,h}^2.$$

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$$|||v_h|||^2_{loc,h} := \|\nabla_x v_h\|^2_Q + \frac{1}{2}\|v_h\|^2_{\Sigma_T} + \sum_{K \in \mathcal{K}_h} \delta_K \, \|\partial_t v_h\|^2_K,$$

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$$a_{loc,h}(u_h, v_h) \geq \mu_{c,loc} \, |||v_h|||^2_{loc,h}.$$

Uniform boundedness of $a_{loc,h}(\cdot,\cdot)$ on $V_{0h,*} \times V_{0h}$

Let $V_{0h,*} = H_0^{\Delta_x,1}(Q) + V_{0h}$ equipped with the norm

$$|\!|\!| v |\!|\!|^2_{loc,h,*} := |\!|\!| v |\!|\!|^2_{loc,h} + \sum_{K \in \mathcal{K}_h} \left(\delta_K^{-1} \|v\|_K^2 + \delta_K \|\Delta_x v\|_K^2 \right).$$

Lemma (boundedness)

Let $\theta_K \in \left(0, \frac{h_K}{d\, C^2_{int,1}}\right]$, $K \in \mathcal{K}_h$. Then, $a_{loc,h}(\cdot,\cdot)$ is **uniformly bounded** on $V_{0h,*} \times V_{0h}$, i.e., there exist a constant $\mu_{b,loc} > 0$ independent on h_K such that

$$|a_{loc,h}(v, v_h)| \leq \mu_{b,loc} \, \|v\|_{loc,h,*} \, \|v_h\|_{loc,h}, \quad \forall v \in V_{0h,*}, \quad \forall v_h \in V_{0h}.$$

Approximation properties and consistency

Lemma (approximation error estimates)

Let $l, s \in \mathbb{N}$ be $1 \leq l \leq s \leq p+1$, and $u \in H^s_{0,\underline{0}}(Q)$. Then, $\exists\, \Pi_h : H^s_{0,\underline{0}}(Q) \rightarrow V_{0h}$ and $C_1, C_2 > 0$, such that **a priori error estimates** hold

$$\|u - \Pi_h u\|^2_{loc,h} \leq C_1 \sum_{K \in \mathcal{K}_h} h_K^{2(s-1)} \sum_{i=0}^{s} c_K^{2\,i}\, |u|^2_{H^i(\underline{K})},$$

$$\|u - \Pi_h u\|^2_{loc,h,*} \leq C_2 \sum_{K \in \mathcal{K}_h} h_K^{2(s-1)} \sum_{i=0}^{s} c_K^{2\,i}\, |u|^2_{H^i(\underline{K})}.$$

where $K \in \mathcal{K}_h$ is the mesh element and $\underline{K}$ is its support extension on the physical domain.

Lemma (consistency)

Let $p \geq 2$. If the solution $u \in H_0^{\Delta_x, 1}(Q)$, then it satisfies the **consistency variational identity**

$$a_{loc,h}(u, v_h) = \ell_{loc,h}(v_h), \qquad \forall v_h \in V_{0h}.$$

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A priori error estimate

Theorem (a priori error estimates)

Let $p \geq 2$, $u \in H_0^{\Delta_x,1}(Q) \cap H_0^s$, $s \geq 2$, be the exact solution, and
$u_h \in V_{0h}$ be a solution of discrete IgA scheme

$$a_{loc,h}(u_h, v_h) = \ell_{loc,h}(v_h), \quad \forall u_h, v_h \in V_{0h} \quad \text{with} \quad \theta_K \in \left(0, \frac{h_K}{d\, C^2_{int,1}}\right], \quad K \in \mathcal{K}_h.$$

Then, the **a priori error estimate**

$$|\!|\!| u - u_h |\!|\!|^2_{loc,h} \leq C \sum_{K \in \mathcal{K}_h} h_K^{2(r-1)} \sum_{i=0}^{r} c_K^{2i} \, |u|^2_{H^i(\underline{K})}, \quad r = \min\{s, p+1\},$$

holds, where p denotes the polynomial degree of the THB-splines,
$C = \left(1 + \frac{\mu_{loc,b}}{\mu_{loc,c}}\right) C_2$ is a constant independent of h_K,
$K \in \mathcal{K}_h$ and its support extension $\underline{K}$, and
$\mu_{loc,b}$ and $\mu_{loc,c}$ are constant in boundedness and coercivity inequalities, respectively.

Adaptive space-time IgA schemes

Majorants for the heat equation with Dirichlet BC

For given $f \in L^2(Q)$ and $u_0 \in H^1_0(\Omega)$, find $u \in H_0^{\Delta_x,1}(Q)$

$$u_t - \Delta_x u = f \text{ in } Q, \quad u = u_D \text{ on } \Sigma, \quad u(0,x) = u_0 \text{ on } \Sigma_0.$$

The error $e = u - v = u - u_h$ is tracked by the norms

$$\|e\|^2_{loc,h} := \|\nabla_x e\|^2_Q + \frac{1}{2}\|e\|^2_{\Sigma_T} + \sum_{K\in\mathcal{K}_h} \delta_K \|\partial_t e\|^2_K, \quad \delta_K = \theta_K h_K, \quad \theta_K \in \left(0, \frac{h_K}{d\, C^2_{int,1}}\right].$$

For any $v \in H_0^{\Delta_x,1}(Q)$ and $\mathbf{y} \in H(Q, \mathrm{div}_x)$, $w \in H_0^{\Delta_x,1}(Q)$, and $\beta, \alpha > 0$, we have a posteriori estimates

$$|||e|||^2 := \|\nabla_x e\|^2_Q + \|e\|^2_{\Sigma_T} \le \overline{\mathrm{M}}^{\mathrm{I},2}(v, \mathbf{y}; \beta) \quad (\overline{\mathrm{M}}^{\mathrm{II},2}(v, \mathbf{y}, w; \beta^{\mathrm{II}}))$$

and error identity

$$\|e\|^2_{\mathcal{L},Q} := \|\Delta_x e\|^2_Q + \|\partial_t e\|^2_Q + \|\nabla_x e\|^2_{\Sigma_T} \equiv \mathbb{E}\mathrm{d}^2(v).$$

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$$|||e|||^2_{\mathcal{L},Q} := \|\Delta_x e\|^2_Q + \|\partial_t e\|^2_Q + \|\nabla_x e\|^2_{\Sigma_T} = \mathrm{Ed}^2(v).$$

Majorants for the heat equation with Dirichlet BC

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For any $v \in H_0^{\Delta_x,1}(Q)$ and $\mathbf{y} \in H(Q, \mathrm{div}_x)$, $w \in H_0^{\Delta_x,1}(Q)$, and $\beta, \alpha > 0$,
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Reconstruction of optimal $\overline{\mathrm{M}}^{\mathrm{I}}(v, \boldsymbol{y}; \beta^{\mathrm{I}})$

Solving $\{\boldsymbol{y}_{\min}, \beta^{\mathrm{I}}_{\min}\} := \arg \inf\limits_{\beta^{\mathrm{I}}>0} \inf\limits_{\boldsymbol{y}\in H(Q,\mathrm{div}_x)} \overline{\mathrm{M}}^{\mathrm{I},2}(v, \boldsymbol{y}; \beta^{\mathrm{I}})$, where

$$\overline{\mathrm{M}}^{\mathrm{I},2}(v, \boldsymbol{y}; \beta^{\mathrm{I}}) := (1+\beta^{\mathrm{I}}) \underbrace{\|\boldsymbol{y} - \nabla_x v\|^2_Q}_{\overline{\mathrm{m}}^{\mathrm{I},2}_{\mathrm{d}}} + \left(1 + \frac{1}{\beta^{\mathrm{I}}}\right) C^2_{\mathrm{F}\Omega} \underbrace{\|f + \mathrm{div}_x \boldsymbol{y} - \partial_t v\|^2_Q}_{\overline{\mathrm{m}}^{\mathrm{I},2}_{\mathrm{eq}}}$$

leads to

- the auxiliary variation problem for the optimal $\boldsymbol{y}_{\min}$, i.e.,

$$\frac{C^2_{\mathrm{F}\Omega}}{\beta^{\mathrm{I}}_{\min}} (\mathrm{div}_x \boldsymbol{y}_{\mathbf{min}}, \mathrm{div}_x \boldsymbol{\eta})_Q + (\boldsymbol{y}_{\mathbf{min}}, \boldsymbol{\eta})_Q = -\frac{C^2_{\mathrm{F}\Omega}}{\beta^{\mathrm{I}}_{\min}} (f - \partial_t v, \mathrm{div}_x \boldsymbol{\eta})_Q + (\nabla_x v, \boldsymbol{\eta})_Q,$$

- with the optimal $\beta^{\mathrm{I}}_{\min} := \frac{C_{\mathrm{F}\Omega}\, \overline{\mathrm{m}}^{\mathrm{I}}_{\mathrm{eq}}}{\overline{\mathrm{m}}^{\mathrm{I}}_{\mathrm{d}}}$.

IgA spaces for u_h approximation

$$\widehat{V}_h \equiv \widehat{\mathcal{S}}_h^p := \mathrm{span}\,\{\widehat{B}_{i,p}\},$$

$$u_h \in V_h \equiv \mathcal{S}_h^p := \{\widehat{V}_h \circ \Phi^{-1}\} \cap H^1_{u_D}(Q) := \mathrm{span}\,\{\phi_{h,i} := \widehat{B}_{i,p} \circ \Phi^{-1}\}_{i \in \mathcal{I}} \cap H^1_{u_D}(Q).$$

Generated approximation u_h is presented as

$$u_h(x) = \sum_{i \in \mathcal{I}} \mathrm{u}_i\, \phi_{h,i}(x), \quad \underline{\mathrm{u}}_h := [\mathrm{u_i}]_{\mathrm{i} \in \mathcal{I}} \in \mathbb{R}^{|\mathcal{I}|},$$

where $\underline{\mathrm{u}}_h$ is a vector of DOFs defined by a system

$$\mathrm{K}_h\, \underline{\mathrm{u}}_h = \mathrm{f}_h, \qquad : t_{\mathrm{as}}(u_h) + t_{\mathrm{sol}}(u_h)$$

$$\mathrm{K}_h := \left[a_{s,h}(\phi_{h,i}, \phi_{h,j})\right]_{i,j}^{\mathcal{I}},$$

$$\mathrm{f}_h := \left[\ell_{s,h}(\phi_{h,i})\right]_{i}^{\mathcal{I}}.$$

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$$\widehat{V}_h \equiv \widehat{\mathcal{S}}_h^p := \mathrm{span}\,\{\widehat{B}_{i,p}\},$$
$$u_h \in V_h \equiv \mathcal{S}_h^p := \{\widehat{V}_h \circ \Phi^{-1}\} \cap H^1_{u_D}(Q) := \mathrm{span}\,\{\phi_{h,i} := \widehat{B}_{i,p} \circ \Phi^{-1}\}_{i \in \mathcal{I}} \cap H^1_{u_D}(Q).$$

Generated approximation u_h is presented as

$$u_h(x) = \sum_{i \in \mathcal{I}} \underline{\mathrm{u}}_i\, \phi_{h,i}(x), \quad \underline{\mathrm{u}}_h := [\mathrm{u}_\mathrm{i}]_{\mathrm{i} \in \mathcal{I}} \in \mathbb{R}^{|\mathcal{I}|},$$

where $\underline{\mathrm{u}}_h$ is a vector of DOFs defined by a system

$$\mathrm{K}_h\, \underline{\mathrm{u}}_h = \mathrm{f}_h, \qquad : t_{\mathrm{as}}(u_h) + t_{\mathrm{sol}}(u_h)$$
$$\mathrm{K}_h := \left[a_{s,h}(\phi_{h,i}, \phi_{h,j})\right]_{i,j}^{\mathcal{I}},$$
$$\mathrm{f}_h := \left[\ell_{s,h}(\phi_{h,i})\right]_{i}^{\mathcal{I}}.$$

IgA spaces for the flux reconstruction $\boldsymbol{y}_h$

$$\widehat{Y}_h \equiv \oplus^{d+1} \widehat{\mathcal{S}}^q_h,$$

$$\boldsymbol{y}_h = \begin{bmatrix} y_h^{(1)} \\ \dots \\ y_h^{(d+1)} \end{bmatrix} \in Y_h \equiv \oplus^{d+1} \mathcal{S}^q_h := \left\{ \widehat{Y}_h \circ \Phi^{-1} \right\} = \mathrm{span}\left\{ \psi_{h,i} := [\widehat{B}_{i,q}]^{d+1} \circ \Phi^{-1} \right\}_{i \in \mathcal{I}}$$

Generated reconstruction of $\boldsymbol{y}_h$ is presented as

$$\boldsymbol{y}_h(x) := \sum_{i \in \mathcal{I} \times (d+1)} \underline{\mathbf{y}}_{h,i} \, \psi_{h,i}(x),$$

where $\underline{\mathbf{y}}_h := \left[\underline{\mathbf{y}}_{h,i} \right]_{i \in \mathcal{I} \times (d+1)} \in \mathbb{R}^{(d+1)|\mathcal{I}|}$ is a vector of DOFs of $\boldsymbol{y}_h$ defined by a system

$$\left(C^2_{\mathrm{F}\Omega} \, \mathrm{Div}_h + \beta \, \mathrm{M}_h \right) \underline{\mathbf{y}}_h = - C^2_{\mathrm{F}\Omega} \, \mathrm{z}_h + \beta \, \mathrm{g}_h, \qquad : t_{\mathrm{as}}(\boldsymbol{y}_h) + t_{\mathrm{sol}}(\boldsymbol{y}_h)$$

with

$$\mathrm{Div_h} := \left[(\mathrm{div}_x \psi_i, \mathrm{div}_x \psi_j) \right]_{i,j=1}^{(d+1)|\mathcal{I}|}, \qquad \mathrm{z}_h := \left[(f - \partial_t v, \mathrm{div}_x \psi_j) \right]_{j=1}^{(d+1)|\mathcal{I}|},$$

$$\mathrm{M_h} := \left[(\psi_i, \psi_j) \right]_{i,j=1}^{(d+1)|\mathcal{I}|}, \qquad \mathrm{g}_h := \left[(\nabla_x v, \psi_j) \right]_{j=1}^{(d+1)|\mathcal{I}|}.$$

IgA spaces for the flux reconstruction $\boldsymbol{y}_h$

$$\widehat{Y}_h \equiv \oplus^{d+1} \widehat{\mathcal{S}}_h^q,$$

$$\boldsymbol{y}_h = \begin{bmatrix} y_h^{(1)} \\ \dots \\ y_h^{(d+1)} \end{bmatrix} \in Y_h \equiv \oplus^{d+1} \mathcal{S}_h^q := \left\{ \widehat{Y}_h \circ \Phi^{-1} \right\} = \mathrm{span}\{\psi_{h,i} := [\widehat{B}_{i,q}]^{d+1} \circ \Phi^{-1}\}_{i \in \mathcal{I}}$$

Generated reconstruction of $\boldsymbol{y}_h$ is presented as

$$\boldsymbol{y}_h(x) := \sum_{i \in \mathcal{I} \times (d+1)} \underline{\mathbf{y}}_{h,i}\, \psi_{h,i}(x),$$

where $\underline{\mathbf{y}}_h := [\underline{\mathbf{y}}_{h,i}]_{i \in \mathcal{I} \times (d+1)} \in \mathbb{R}^{(d+1)|\mathcal{I}|}$ is a vector of DOFs of $\boldsymbol{y}_h$ defined by a system

$$\left(C_{\mathrm{F}\Omega}^2\, \mathrm{Div}_h + \beta\, \mathrm{M}_h\right) \underline{\mathbf{y}}_h = -C_{\mathrm{F}\Omega}^2\, \mathrm{z}_h + \beta\, \mathrm{g}_h, \qquad : t_{\mathrm{as}}(y_h) + t_{\mathrm{sol}}(y_h)$$

with

$$\mathrm{Div_h} := \left[(\mathrm{div}_x \psi_i, \mathrm{div}_x \psi_j)\right]_{i,j=1}^{(d+1)|\mathcal{I}|}, \qquad \mathrm{z}_h := \left[(f - \partial_t v, \mathrm{div}_x \psi_j)\right]_{j=1}^{(d+1)|\mathcal{I}|},$$

$$\mathrm{M_h} := \left[(\psi_i, \psi_j)\right]_{i,j=1}^{(d+1)|\mathcal{I}|}, \qquad \mathrm{g}_h := \left[(\nabla_x v, \psi_j)\right]_{j=1}^{(d+1)|\mathcal{I}|}.$$

Single refinement step for u_h approximation

Input: $\mathcal{K}_h$ {discretization of Q}, $\mathrm{span}\,\{\phi_{h,i}\}$, $i = 1, ..., |\mathcal{I}|$ {V_h-basis}

APPROXIMATE:

- compute u_h: ASSEMBLE and SOLVE $\mathrm{K}_h\, \underline{\mathrm{u}}_h = \mathrm{f}_h$:$t_{\mathrm{as}}(u_h) + t_{\mathrm{sol}}(u_h)$

Evaluate $e = u - u_h$ in terms of $\|e\|$, $\|e\|_{loc,h}$, and $\|e\|_{\mathcal{L}}$

ESTIMATE:

- compute $\overline{\mathrm{M}}^{\mathrm{I}}(u_h, \boldsymbol{y}_h)$:$t_{\mathrm{as}}(\boldsymbol{y}_h) + t_{\mathrm{sol}}(\boldsymbol{y}_h)$
- compute $\overline{\mathrm{M}}^{\mathrm{II}}(u_h, \boldsymbol{y}_h, w_h)$:$t_{\mathrm{as}}(w_h) + t_{\mathrm{sol}}(w_h)$
- compute $\mathbb{E}\mathrm{d}(u_h)$

MARK: Using marking $\mathbb{M}_{\mathrm{BULK}}(\sigma)$, select elements K of mesh $\mathcal{K}_h$ that must be refined

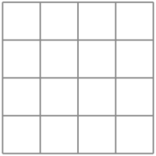
REFINE: Execute the refinement strategy $\mathcal{K}_{h_{ref}} = R(\mathcal{K}_h)$

Output: $\mathcal{K}_{h_{ref}}$ {refined discretization of Q}

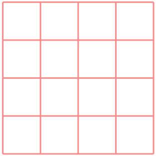
Choice of B-Splines (THB-Splines) for $\boldsymbol{y}_h$ and w_h

We use the idea from [Kleiss, Tomar, 2015]:

- $u_h \in V_h \equiv \mathcal{S}_h^p$
- u_h is approx. on $\mathcal{K}_h$



- $\boldsymbol{y}_h \in Y_{Mh} \equiv \oplus^{d+1} \mathcal{S}_{Mh}^q$
- $q \gg p$,
- $\boldsymbol{y}_{Mh}$ is reconstructed on $\mathcal{K}_{Mh}$, $M \in \mathbb{N}^+$



- $w_h \in W_{Lh} \equiv \mathcal{S}_{Lh}^r$
- $r \gg p$,
- w_{Lh} is reconstructed on $\mathcal{K}_{Lh}$, $L \in \mathbb{N}^+$

Choice of B-Splines (THB-Splines) for $\boldsymbol{y}_h$ and w_h

We use the idea from [Kleiss, Tomar, 2015]:

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- u_h is approx. on $\mathcal{K}_h$

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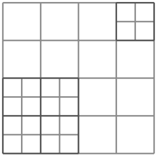
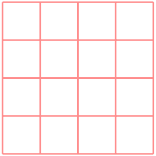
- $q \gg p$,
- $\boldsymbol{y}_{Mh}$ is reconstructed on $\mathcal{K}_{Mh}$, $M \in \mathbb{N}^+$

- $w_h \in W_{Lh} \equiv \mathcal{S}_{Lh}^r$
- $r \gg p$,
- w_{Lh} is reconstructed on $\mathcal{K}_{Lh}$, $L \in \mathbb{N}^+$





Choice of B-Splines (THB-Splines) for $\boldsymbol{y}_h$ and w_h

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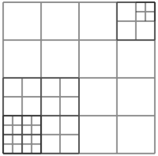
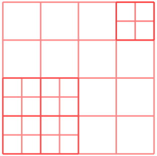
- $u_h \in V_h \equiv \mathcal{S}_h^p$
- u_h is approx. on $\mathcal{K}_h$

- $\boldsymbol{y}_h \in Y_{Mh} \equiv \oplus^{d+1} \mathcal{S}_{Mh}^q$
- $q \gg p$,
- $\boldsymbol{y}_{Mh}$ is reconstructed on $\mathcal{K}_{Mh}$, $M \in \mathbb{N}^+$

- $w_h \in W_{Lh} \equiv \mathcal{S}_{Lh}^r$
- $r \gg p$,
- w_{Lh} is reconstructed on $\mathcal{K}_{Lh}$, $L \in \mathbb{N}^+$





Choice of B-Splines (THB-Splines) for $\boldsymbol{y}_h$ and w_h

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- $q \gg p$,
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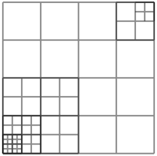
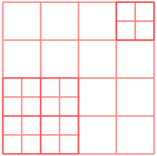
- $r \gg p$,
- w_{Lh} is reconstructed on $\mathcal{K}_{Lh}$, $L \in \mathbb{N}^+$





Numerical results

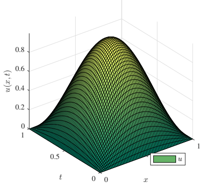
Example 1

Given data of 1d+t dimensional problem:

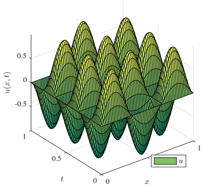
- $\Omega = (0, 1), \; T = 1$
- $u = \sin(k_1 \, \pi \, x) \, \sin(k_2 \, \pi \, t)$
- $f = \sin(k_1 \pi x) \left(k_2 \pi \cos(k_2 \pi t) + k_1^2 \pi^2 \sin(k_2 \pi t) \right)$
- $u_D = 0$

Discretization:

- $u_h \in S_h^2$ and $u_h \in S_h^3$
- **Example 1-1:** $k_1 = k_2 = 1$:
- **Example 1-2:** $k_1 = 3, \; k_2 = 6$:



$k_1 = k_2 = 1$



$k_1 = 3, \; k_2 = 6$

Example 1-1. Adaptive refinement for $u_h \in S_h^2$ and $u_h \in S_h^3$

# ref.	$\|\|\|e\|\|\|_Q$	$I_{\rm eff}(\overline{\rm M}^{\rm I})$	$I_{\rm eff}(\overline{\rm M}^{\rm II})$	$\|\|\|e\|\|\|_{loc,h}$	$\|\|\|e\|\|\|_{\mathcal{L}}$	$I_{\rm eff}(\mathbb{E}\rm d)$	e.o.c. ($\|\|\|e\|\|\|_{loc,h}$)	e.o.c. ($\|\|\|e\|\|\|_{\mathcal{L}}$)
(a) $u_h \in S_h^2$, $y_h \in \oplus^2 S_{7h}^4$, and $w_h \in S_{7h}^4$								
2	2.9034e-03	1.94	1.17	3.0649e-03	2.9197e-01	1.00	2.38	1.40
4	3.3878e-04	3.14	1.33	3.5057e-04	9.3154e-02	1.00	1.96	1.07
8	9.2649e-06	5.78	3.23	9.2835e-06	1.7351e-02	1.00	3.79	1.79
(b) $u_h \in S_h^3$, $y_h \in \oplus^2 S_{5h}^6$, and $w_h \in S_{5h}^6$								
2	4.9924e-03	1.31	1.04	5.0700e-03	1.1918e-01	1.00	5.08	4.18
4	1.3562e-04	1.64	1.30	1.3591e-04	8.9725e-03	1.00	3.56	2.89
8	3.5507e-07	3.44	1.24	3.5535e-07	1.6376e-04	1.00	3.11	2.13

Efficiency of $\overline{\rm M}^{\rm I}$, $\overline{\rm M}^{\rm II}$, and $\mathbb{E}\rm d$ for $\sigma = 0.4$ ($N_{\rm ref,0} = 3$).

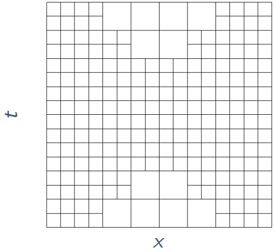
Example 1-1. Adaptive refinement for $u_h \in S_h^2$ and $u_h \in S_h^3$

# ref.	d.o.f. u_h	d.o.f. y_h	d.o.f. w_h	t_{as} u_h	t_{as} y_h	t_{as} w_h	t_{sol} u_h	t_{sol} y_h	t_{sol} w_h	$\frac{t_{appr.}}{t_{er.est.}}$
(a) $u_h \in S_h^2$, $y_h \in \oplus^2 S_{7h}^4$, and $w_h \in S_{7h}^4$										
6	12935	288	144	1.55e+01	3.97e-01	3.83e-01	2.17e+00	2.30e-03	1.37e-03	44.25
7	34037	288	144	4.90e+01	3.98e-01	3.73e-01	9.58e+00	3.36e-03	1.42e-03	145.95
8	61258	288	144	9.37e+01	3.80e-01	3.62e-01	2.42e+01	2.10e-03	1.83e-03	308.55
				$t_{as}(u_h)$:	$t_{as}(y_h)$:	$t_{as}(w_h)$	$t_{sol}(u_h)$:	$t_{sol}(y_h)$:	$t_{sol}(w_h)$	
				258.63	1.05	1.00	**13252.51**	1.15	1.00	
(b) $u_h \in S_h^3$, $y_h \in \oplus^2 S_{5h}^5$, and $w_h \in S_{5h}^5$										
6	13742	338	169	1.62e+01	7.03e-01	7.03e-01	2.11e+00	2.53e-03	1.43e-03	25.95
7	35091	644	322	5.36e+01	5.65e+00	5.52e+00	1.10e+01	9.31e-03	5.29e-03	11.41
8	78561	744	372	1.91e+02	5.61e+00	5.03e+00	2.40e+01	2.51e-02	7.56e-03	38.15
				$t_{as}(u_h)$:	$t_{as}(y_h)$:	$t_{as}(w_h)$	$t_{sol}(u_h)$:	$t_{sol}(y_h)$:	$t_{sol}(w_h)$	
				37.97	1.11	1.00	**3168.34**	3.31	1.00	

Assembling and solving time spent for the systems defining d.o.f. of u_h, y_h, and w_h.

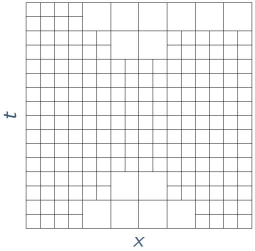
Example 1-1. Comparison of meshes for $\mathbb{M}_{\mathrm{BULK}}(0.4)$

ref. based on
true error $\|u - u_h\|^2_{loc,h,K}$



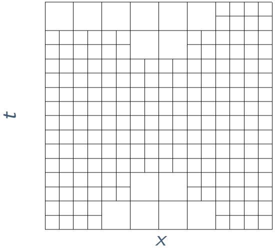
ref. 1

ref. based on
true error $|\!|\!|u - u_h|\!|\!|^2_K$



ref. 1

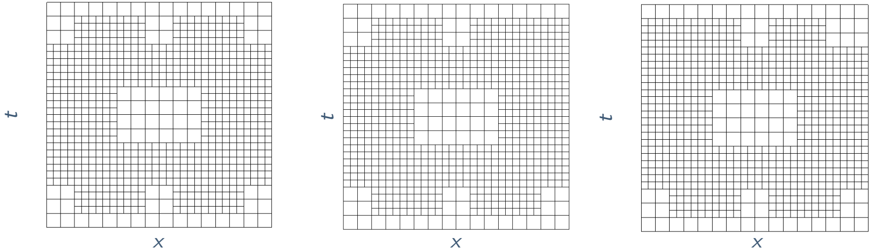
ref. based on
indicator $\|\mathbf{y}_h - \nabla_x u_h\|^2_K$



ref. 1

Example 1-1. Comparison of meshes for $\mathbb{M}_{\mathrm{BULK}}(0.4)$

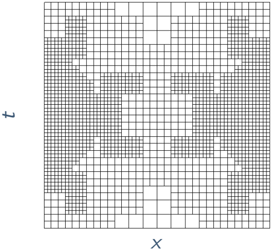
ref. based on true error $\|u - u_h\|^2_{loc,h,K}$	ref. based on true error $\|\|\|u - u_h\|\|\|^2_K$	ref. based on **indicator** $\|\boldsymbol{y}_h - \nabla_x u_h\|^2_K$



ref. 2 ref. 2 ref. 2

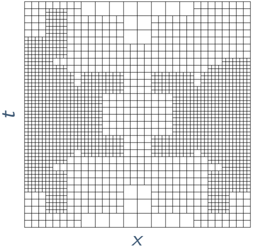
Example 1-1. Comparison of meshes for $\mathbb{M}_{\mathrm{BULK}}(0.4)$

ref. based on
true error $\|u - u_h\|^2_{loc,h,K}$



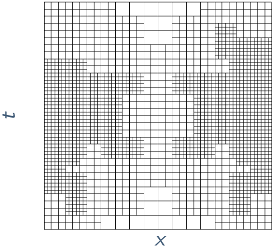
ref. 3

ref. based on
true error $|||u - u_h|||^2_K$



ref. 3

ref. based on
indicator $\|\boldsymbol{y}_h - \nabla_x u_h\|^2_K$



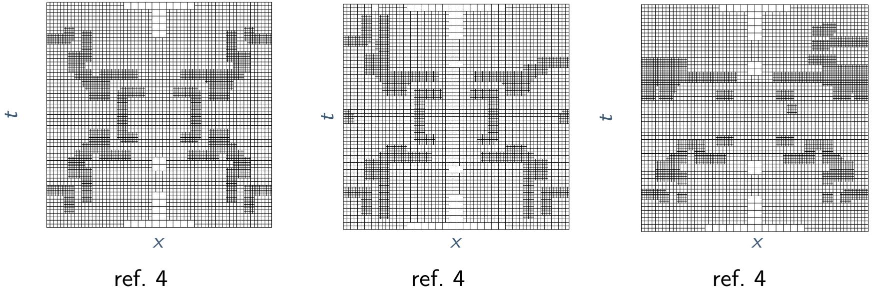
ref. 3

Example 1-1. Comparison of meshes for $\mathbb{M}_{\mathrm{BULK}}(0.4)$

ref. based on **true error** $\|u - u_h\|^2_{loc,h,K}$

ref. based on **true error** $|\!|\!|u - u_h|\!|\!|^2_K$

ref. based on **indicator** $\|\boldsymbol{y}_h - \nabla_x u_h\|^2_K$

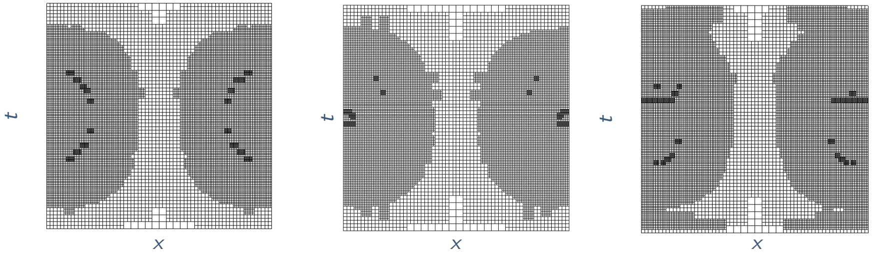


Example 1-1. Comparison of meshes for $\mathbb{M}_{\mathrm{BULK}}(0.4)$

ref. based on true error $\|u - u_h\|^2_{loc,h,K}$

ref. based on true error $|\!|\!|u - u_h|\!|\!|^2_K$

ref. based on **indicator** $\|\boldsymbol{y}_h - \nabla_x u_h\|^2_K$



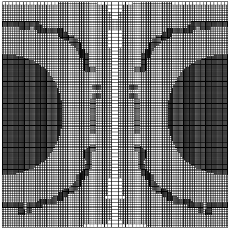
ref. 5

ref. 5

ref. 5

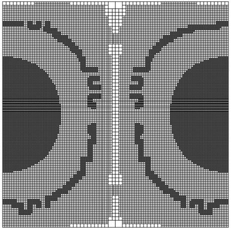
Example 1-1. Comparison of meshes for $\mathbb{M}_{\mathrm{BULK}}(0.4)$

ref. based on
true error $\|u - u_h\|^2_{loc,h,K}$



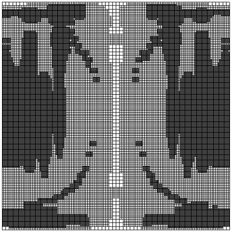
ref. 6

ref. based on
true error $\|\!|u - u_h|\!\|^2_K$



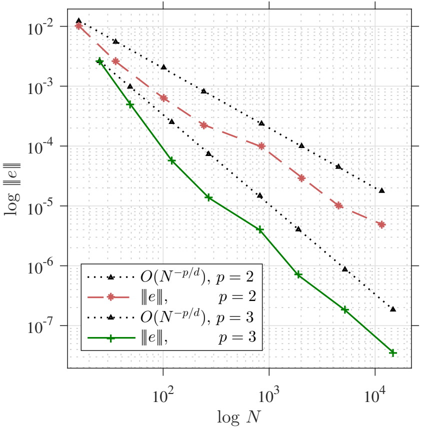
ref. 6

ref. based on
indicator $\|\boldsymbol{y}_h - \nabla_x u_h\|^2_K$



ref. 6

Example 1-1. Error order of convergence



The e.o.c. for $k_1 = k_2 = 1$.

Example 1-2. Adaptive refinement, $u_h \in S_h^2$, $\boldsymbol{y}_h \in \oplus^2 S_{5h}^7$, and $w_h \in S_{5h}^7$

# ref.	$\|\!\|\!\| e \|\!\|\!\|_Q$	$I_{\mathrm{eff}}(\overline{\mathrm{M}}^{\mathrm{I}})$	$I_{\mathrm{eff}}(\overline{\mathrm{M}}^{\mathrm{II}})$	$\|\!\|\!\| e \|\!\|\!\|_{loc,h}$	$\|\!\|\!\| e \|\!\|\!\|_{\mathcal{L}}$	$I_{\mathrm{eff}}(\mathrm{Ed})$	e.o.c. ($\|\!\|\!\| e \|\!\|\!\|_{loc,h}$)	e.o.c. ($\|\!\|\!\| e \|\!\|\!\|_{\mathcal{L}}$)
(a) $\mathbb{M}_{\mathrm{BULK}}(0.4)$								
2	5.7161e-01	2.11	1.38	5.7163e-01	6.2371e+01	1.00	2.99	1.19
3	1.3927e-01	5.77	2.20	1.3928e-01	3.1026e+01	1.00	2.30	1.14
8	1.2298e-03	1.44	1.16	1.2298e-03	2.6917e+00	1.00	5.60	2.30
(b) $\mathbb{M}_{\mathrm{BULK}}(0.6)$								
2	5.7161e-01	2.11	1.38	5.7163e-01	6.2371e+01	1.00	2.99	1.19
3	1.7942e-01	4.69	1.96	1.7945e-01	3.2971e+01	1.00	2.18	1.20
8	2.7492e-03	1.44	1.15	2.7492e-03	4.0721e+00	1.00	4.75	1.91

Efficiency of $\overline{\mathrm{M}}^{\mathrm{I}}$, $\overline{\mathrm{M}}^{\mathrm{II}}$, and Ed for $\mathbb{M}_{\mathrm{BULK}}(0.4)$ and $\mathbb{M}_{\mathrm{BULK}}(0.6)$ ($N_{\mathrm{ref},0} = 3$).

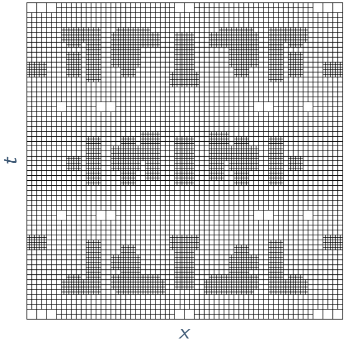
Example 1-2. Adaptive refinement, $u_h \in S_h^2$, $\boldsymbol{y}_h \in \oplus^2 S_{5h}^7$, and $w_h \in S_{5h}^7$

# ref.	d.o.f. u_h	d.o.f. $\boldsymbol{y}_h$	d.o.f. w_h	$t_{\rm as}$ u_h	$t_{\rm as}$ $\boldsymbol{y}_h$	$t_{\rm as}$ w_h	$t_{\rm sol}$ u_h	$t_{\rm sol}$ $\boldsymbol{y}_h$	$t_{\rm sol}$ w_h	$\frac{t_{\rm appr.}}{t_{\rm er.est.}}$
(a) $\mathbb{M}_{\rm BULK}(0.4)$										
6	30101	450	225	5.99e+01	2.29e+00	2.92e+00	3.57e+00	8.52e-03	4.33e-03	12.14
7	86849	1058	529	3.57e+02	9.30e+00	9.41e+00	1.11e+01	5.19e-02	3.47e-02	19.58
8	141987	2850	1425	6.36e+02	6.50e+01	5.91e+01	2.56e+01	3.00e-01	1.29e-01	5.31
				$t_{\rm as}(u_h)$:	$t_{\rm as}(\boldsymbol{y}_h)$:	$t_{\rm as}(w_h)$	$t_{\rm sol}(u_h)$:	$t_{\rm sol}(\boldsymbol{y}_h)$:	$t_{\rm sol}(w_h)$	
				10.76	1.10	1.00	**198.84**	2.32	1.00	
(b) $\mathbb{M}_{\rm BULK}(0.6)$										
6	15436	450	225	2.61e+01	2.36e+00	2.41e+00	1.77e+00	1.45e-02	3.12e-03	11.73
7	35745	1058	529	8.99e+01	9.86e+00	1.01e+01	4.68e+00	7.06e-02	4.12e-02	9.52
8	52453	2498	1249	1.05e+02	8.03e+01	7.08e+01	7.38e+00	3.47e-01	1.66e-01	1.39
				$t_{\rm as}(u_h)$:	$t_{\rm as}(\boldsymbol{y}_h)$:	$t_{\rm as}(w_h)$	$t_{\rm sol}(u_h)$:	$t_{\rm sol}(\boldsymbol{y}_h)$:	$t_{\rm sol}(w_h)$	
				1.49	1.13	1.00	**44.46**	2.09	1.00	

Assembling and solving time spent for the systems defining d.o.f. of u_h, $\boldsymbol{y}_h$, and w_h.

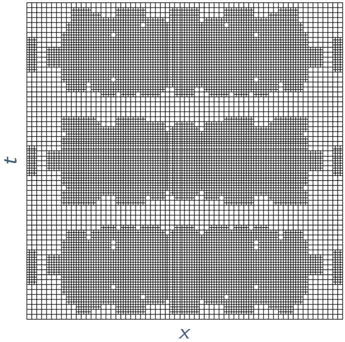
Example 1-2. Comparison of meshes for $\mathbb{M}_{\mathrm{BULK}}(0.6)$ and $\mathbb{M}_{\mathrm{BULK}}(0.4)$

mesh obtained with criterion $\mathbb{M}_{\mathrm{BULK}}(0.6)$



ref. 4

mesh obtained with criterion $\mathbb{M}_{\mathrm{BULK}}(0.4)$

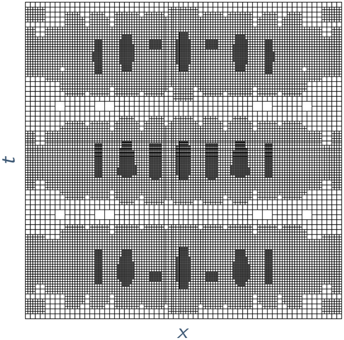


ref. 4

$u_h \in S_h^2$, $\boldsymbol{y}_h \in \oplus^2 S_{5h}^7$, and $w_h \in S_{5h}^7$.

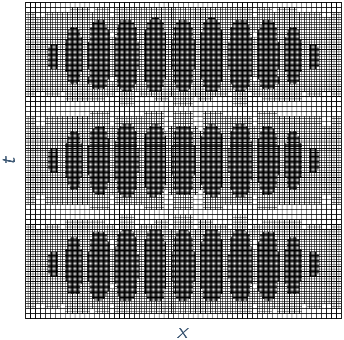
Example 1-2. Comparison of meshes for $\mathbb{M}_{\mathrm{BULK}}(0.6)$ and $\mathbb{M}_{\mathrm{BULK}}(0.4)$

mesh obtained with criterion $\mathbb{M}_{\mathrm{BULK}}(0.6)$



ref. 5

mesh obtained with criterion $\mathbb{M}_{\mathrm{BULK}}(0.4)$

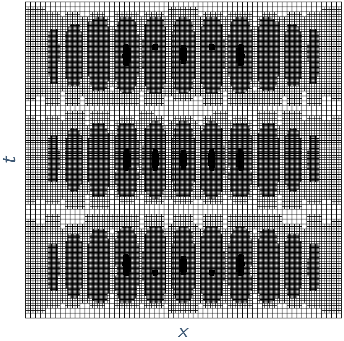


ref. 5

$u_h \in S_h^2$, $\boldsymbol{y}_h \in \oplus^2 S_{5h}^7$, and $w_h \in S_{5h}^7$.

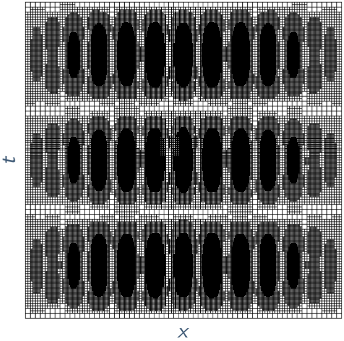
Example 1-2. Comparison of meshes for $\mathbb{M}_{\mathrm{BULK}}(0.6)$ and $\mathbb{M}_{\mathrm{BULK}}(0.4)$

mesh obtained with criterion $\mathbb{M}_{\mathrm{BULK}}(0.6)$



ref. 6

mesh obtained with criterion $\mathbb{M}_{\mathrm{BULK}}(0.4)$

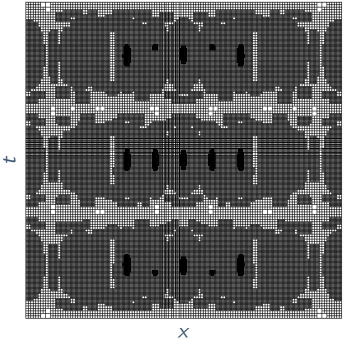


ref. 6

$u_h \in S_h^2$, $\boldsymbol{y}_h \in \oplus^2 S_{5h}^7$, and $w_h \in S_{5h}^7$.

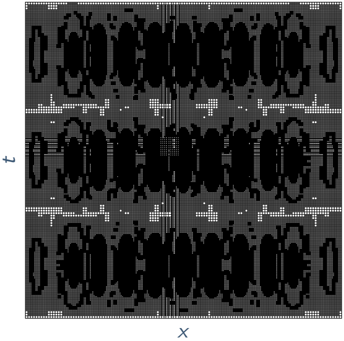
Example 1-2. Comparison of meshes for $\mathbb{M}_{\mathrm{BULK}}(0.6)$ and $\mathbb{M}_{\mathrm{BULK}}(0.4)$

mesh obtained with criterion $\mathbb{M}_{\mathrm{BULK}}(0.6)$



ref. 7

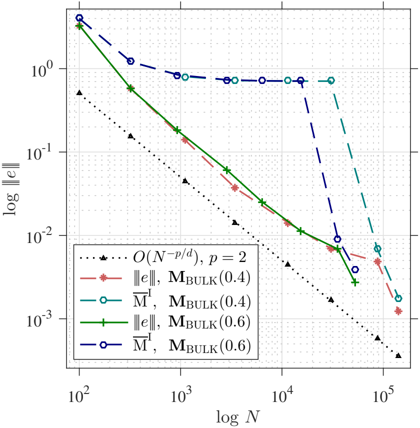
mesh obtained with criterion $\mathbb{M}_{\mathrm{BULK}}(0.4)$



ref. 7

$u_h \in S_h^2$, $\boldsymbol{y}_h \in \oplus^2 S_{5h}^7$, and $w_h \in S_{5h}^7$.

Example 1-2. Error order of convergence



The e.o.c. for $k_1 = 6$, $k_2 = 3$.

Example 2. Moving spatial domains

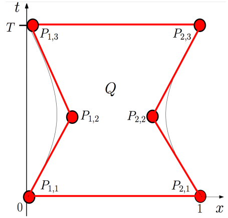
- $Q := \{(x,t)\mathbb{R}^{\mathrm{d}+1} : x \in \Omega(t),\ t \in (0,T)\}$, where
  $\Omega(t) = \{x \in \mathbb{R}^{\mathrm{d}} : a(t) < x < b(t)\},\ t \in (0,T)$
  $a(t) = 0.5\, t\,(1-t)$,
  $b(t) = 1 - a(t)$, and
  $T = 1$

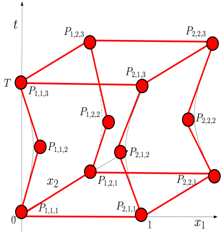
1d+t:

- $u(x,t) = \sin(\pi\, x)\, \sin(\pi\, t)$,
- $f(x,t) = \pi\, \sin(\pi\, x)\, (\cos(\pi\, t) + \pi\, \sin(\pi\, t))$

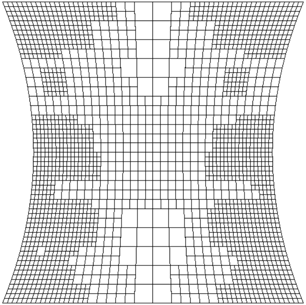
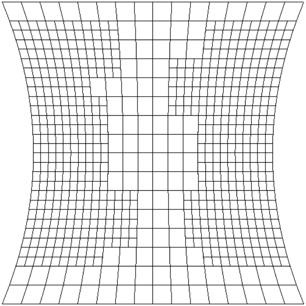
2d+t:

- $u(x,t) = \sin(\pi\, x)\, \sin(\pi\, y)\, \sin(\pi\, t)$,
- $f(x,t) =$
  $(\pi\, \sin(\pi\, x)\, \sin(\pi\, y))\, (\cos(\pi\, t) + 2\,\pi\, \sin(\pi\, t))$





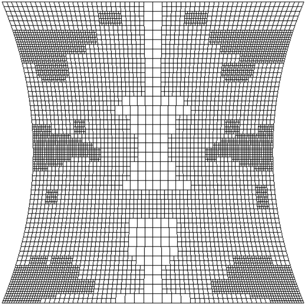
Example 2-1. Mesh refinement for $\mathbb{M}_{\mathrm{BULK}}(0.4)$ and $\mathbb{M}_{\mathrm{BULK}}(0.6)$

$\mathbb{M}_{\mathrm{BULK}}(0.4)$	$\mathbb{M}_{\mathrm{BULK}}(0.6)$
	
ref. 5: $\mathcal{K}_h$	ref. 5: $\mathcal{K}_h$

$u_h \in S_h^2$, $\boldsymbol{y}_h \in \oplus^2 S_{4h}^4$, and $w_h \in S_{4h}^4$.

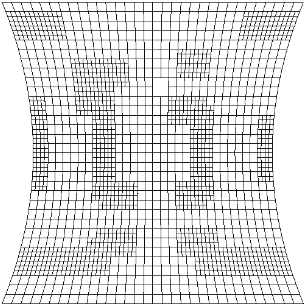
Example 2-1. Mesh refinement for $\mathbb{M}_{\mathrm{BULK}}(0.4)$ and $\mathbb{M}_{\mathrm{BULK}}(0.6)$

$\mathbb{M}_{\mathrm{BULK}}(0.4)$



ref. 6: $\mathcal{K}_h$

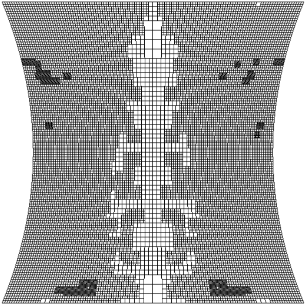
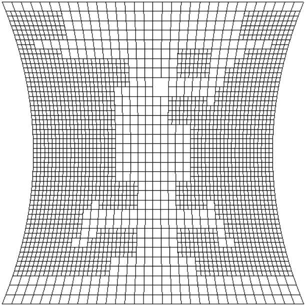
$\mathbb{M}_{\mathrm{BULK}}(0.6)$



ref. 6: $\mathcal{K}_h$

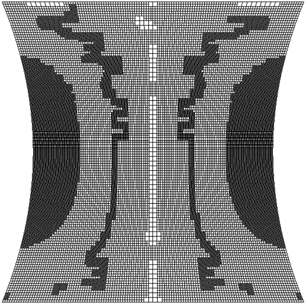
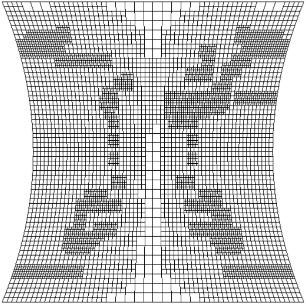
$u_h \in S_h^2$, $\boldsymbol{y}_h \in \oplus^2 S_{4h}^4$, and $w_h \in S_{4h}^4$.

Example 2-1. Mesh refinement for $\mathbb{M}_{\mathrm{BULK}}(0.4)$ and $\mathbb{M}_{\mathrm{BULK}}(0.6)$

$\mathbb{M}_{\mathrm{BULK}}(0.4)$	$\mathbb{M}_{\mathrm{BULK}}(0.6)$
	
ref. 7: $\mathcal{K}_h$	ref. 7: $\mathcal{K}_h$

$u_h \in S^2_h$, $\mathbf{y}_h \in \oplus^2 S^4_{4h}$, and $w_h \in S^4_{4h}$.

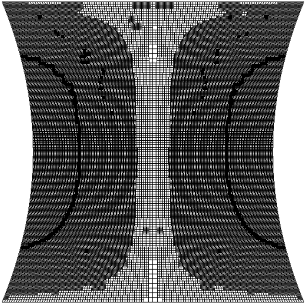
Example 2-1. Mesh refinement for $\mathbb{M}_{\mathrm{BULK}}(0.4)$ and $\mathbb{M}_{\mathrm{BULK}}(0.6)$

$\mathbb{M}_{\mathrm{BULK}}(0.4)$	$\mathbb{M}_{\mathrm{BULK}}(0.6)$
	
ref. 8: $\mathcal{K}_h$	ref. 8: $\mathcal{K}_h$

$u_h \in S^2_h$, $\boldsymbol{y}_h \in \oplus^2 S^4_{4h}$, and $w_h \in S^4_{4h}$.

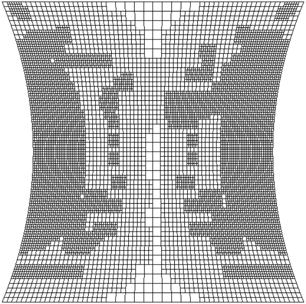
Example 2-1. Mesh refinement for $\mathbb{M}_{\mathrm{BULK}}(0.4)$ and $\mathbb{M}_{\mathrm{BULK}}(0.6)$

$\mathbb{M}_{\mathrm{BULK}}(0.4)$



ref. 9: $\mathcal{K}_h$

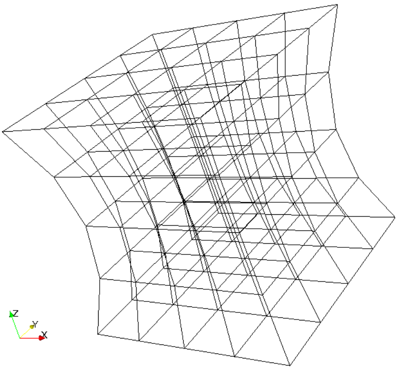
$\mathbb{M}_{\mathrm{BULK}}(0.6)$



ref. 9: $\mathcal{K}_h$

$u_h \in S_h^2$, $\boldsymbol{y}_h \in \oplus^2 S_{4h}^4$, and $w_h \in S_{4h}^4$.

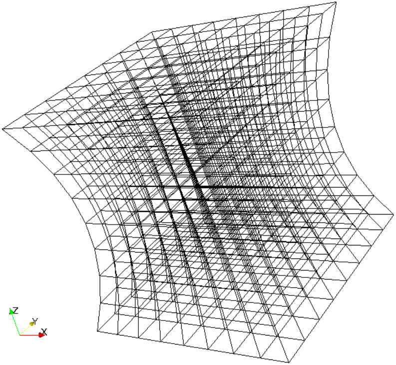
Example 2-2. Mesh refinement for $\mathbb{M}_{\mathrm{BULK}}(0.4)$



ref. 1: $\mathcal{K}_h$

$u_h \in S_h^2$, $\mathbf{y}_h \in \oplus^2 S_{3h}^4$, and $w_h \in S_{3h}^4$.

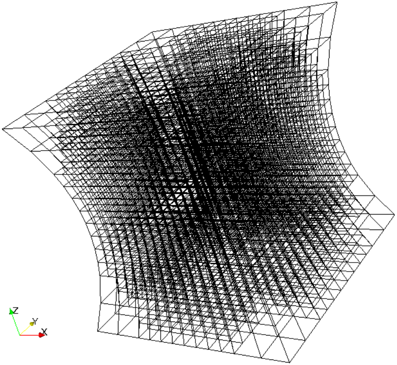
Example 2-2. Mesh refinement for $\mathbb{M}_{\mathrm{BULK}}(0.4)$



ref. 2: $\mathcal{K}_h$

$u_h \in S_h^2$, $\boldsymbol{y}_h \in \oplus^2 S_{3h}^4$, and $w_h \in S_{3h}^4$.

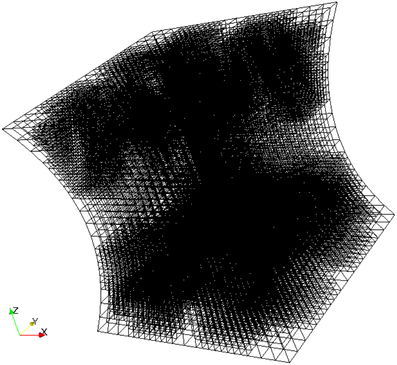
Example 2-2. Mesh refinement for $\mathbb{M}_{\mathrm{BULK}}(0.4)$



ref. 3: $\mathcal{K}_h$

$u_h \in S_h^2$, $\boldsymbol{y}_h \in \oplus^2 S_{3h}^4$, and $w_h \in S_{3h}^4$.

Example 2-2. Mesh refinement for $\mathbb{M}_{\mathrm{BULK}}(0.4)$



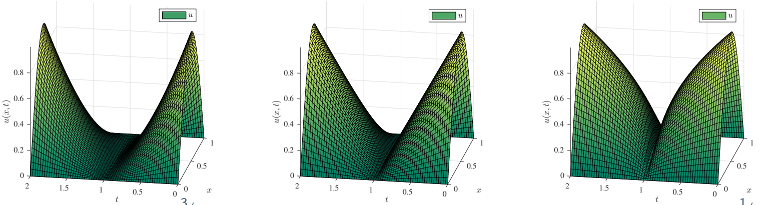
ref. 4: $\mathcal{K}_h$

$u_h \in S_h^2$, $\boldsymbol{y}_h \in \oplus^2 S_{3h}^4$, and $w_h \in S_{3h}^4$.

Example 3: Robustness to problems different singularities

Given data:

- $\Omega = (0, 1)$, $T = 2$
- $u(x, t) = \sin \pi x \, (1 - t)^{\lambda}$, $\lambda = \left\{ \frac{3}{2}, 1, \frac{1}{2} \right\}$, $u \in H^{s, \lambda + \frac{1}{2} - \varepsilon}(Q)$, where $\varepsilon > 0$ is arbitrary small, $s \geq 2$



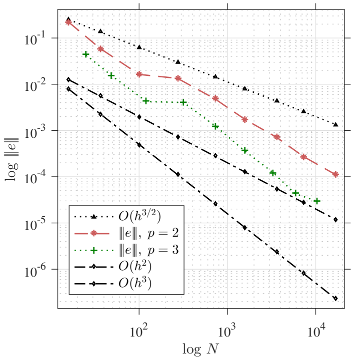
(a) $u = \sin \pi x \, (1 - t)^{3/2}$ (b) $u = \sin \pi x \, |1 - t|$ (c) $u = \sin \pi x \, (1 - t)^{1/2}$

Discretisation:

- $u_h \in S_h^2$
- $\mathbf{y}_h \in \oplus^2 S_h^3$ and $w_h \in S_h^3$

Example 3: Error order of convergence, $\lambda = \frac{3}{2}$

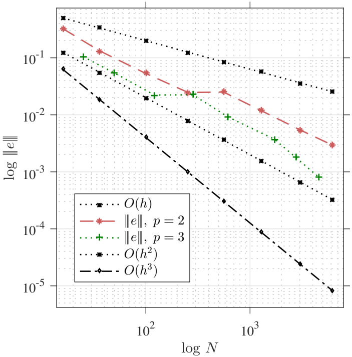
Theoretical (expected) rate $O(h^{3/2})$:



Error order of convergence for for approximations with $u \in S_h^2$ and $u \in S_h^3$.

Example 3: Error order of convergence, $\lambda = 1$

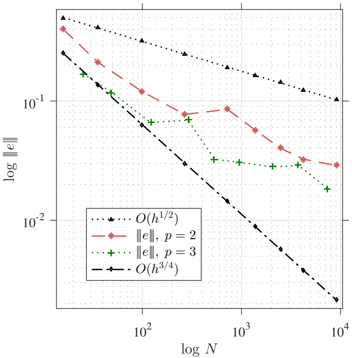
Theoretical (expected) rate $O(h)$:



Error order of convergence for for approximations with $u \in S_h^2$ and $u \in S_h^3$.

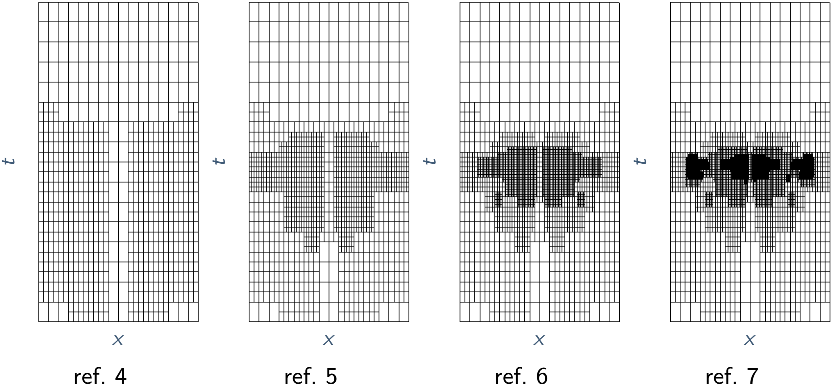
Example 3: Error order of convergence, $\lambda = \frac{1}{2}$

Theoretical (expected) rate $O(h^{1/2})$:



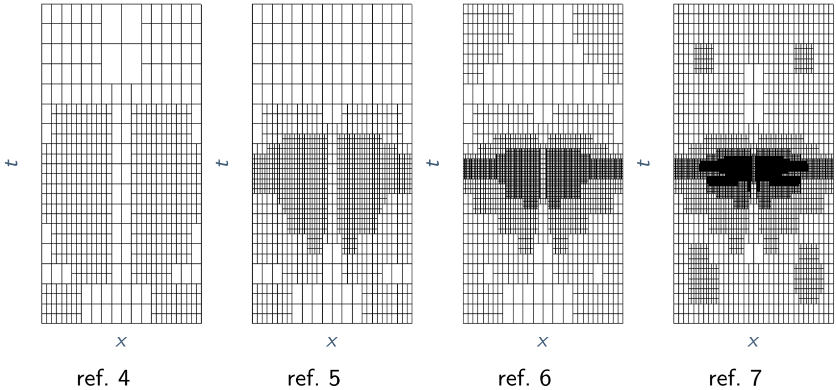
Error order of convergence for for approximations with $u \in S_h^2$ and $u \in S_h^3$.

Example 3: Mesh refinement with $\mathbb{M}_{\mathrm{BULK}}(0.4)$, $\lambda = \frac{3}{2}$



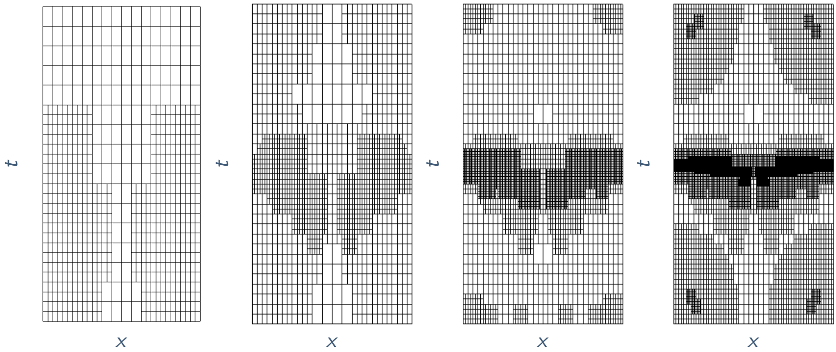
Meshes obtained on the refinement steps 4–7 for $u_h \in S_h^2$.

Example 3: Mesh refinement with $\mathbb{M}_{\mathrm{BULK}}(0.4)$, $\lambda = 1$



Meshes obtained on the refinement steps 4–7 for $u_h \in S_h^2$.

Example 3: Mesh refinement with $\mathbb{M}_{\mathrm{BULK}}(0.4)$, $\lambda = \frac{1}{2}$



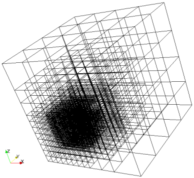
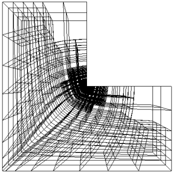
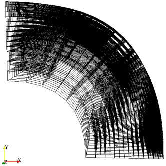
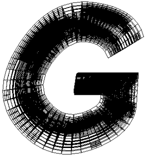
ref. 4 ref. 5 ref. 6 ref. 7

Meshes obtained on the refinement steps 4–7 for $u_h \in S_h^2$.

Conclusions and roadmap

Conclusions and roadmap

- From globally to locally stabalized space-time IgA schemes:
  1. a priori discretization error estimates
  2. functional a posteriori discretization error estimates
  3. adaptive IgA schemes based on global flux reconstruction
- Adaptivity + Fast (multilevel) solvers + Parallelization
  1. improving assembling time, in particular, for the THB-splines
  2. fast solvers for the system providing optimal flux used in the majorant

THANK YOU FOR YOUR ATTENTION!

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U. Langer, S. Matculevich, and S. Repin. Guaranteed error control bounds for the stabilized space-time IgA approximations to parabolic problems, arXiv.org, math.NA/1712.06017, 2017 (submitted).

U. Langer, S. Matculevich, and S. Repin. Functional type error control for Stabilized space- time IgA approximations to parabolic problems. In I. Lirkov and S. Margenov, editors, Large-Scale Scientific Computing (LSSC 2017), Lecture Notes in Computer Science (LNCS), p. 57–66, Springer-Verlag, 2017.

S. Matculevich. Functional approach to the error control in adaptive IgA schemes for elliptic boundary value problems Journal of Computational and Applied Mathematics, doi.org/10.1016/j.cam.2018.05.029, 2018.

U. Langer, S. Matculevich, and S. Repin. A posteriori error estimates for space-time IgA approximations to parabolic initial boundary value problems, arXiv.org, math.NA/1612.08998, 2016.

Example 1. Comparison of different approaches for flux reconstructions

# ref.	$\|\|\|e\|\|\|_Q$	$I_{\rm eff}(\overline{\rm M}^{\rm I})$	e.o.c. ($\|\|\|e\|\|\|_{s,h}$)
uniform refinement			expected $O(h^2)$
(a) majorant with $\boldsymbol{y}_h := \mathrm{argmin}_{\boldsymbol{y}_h \in Y_h} \overline{\rm M}$			
2	2.5516e-03	1.07	3.44
4	1.5947e-04	1.39	2.36
6	9.9670e-06	1.00	2.09
8	6.2294e-07	1.01	2.02
(b) majorant with $\boldsymbol{y}_h = \nabla_x u_h$ (implies residual-type estimate)			
2	2.5516e-03	4.52	3.44
4	1.5947e-04	1.91	2.36
6	9.9670e-06	4.09	2.09
8	6.2294e-07	6.39	2.02
(c) majorant with equilibrated fluxes (implies equalibration-type estimate)			
2	2.5516e-03	36.22	3.44
4	1.5947e-04	869.73	2.36
7	2.4918e-06	138992.85	2.05
8	6.2294e-07	247643.30	2.02

Efficiency of $\overline{\rm M}^{\rm I}$ w.r.t. three approaches of the $\boldsymbol{y}_h \in \oplus^2 S^3_{5h}$ reconstruction.

Example 1. Comparison of different approaches for flux reconstructions

# ref.	$\vert\!\vert\!\vert e \vert\!\vert\!\vert_Q$	$I_{\mathrm{eff}}(\overline{\mathrm{M}}^{\mathrm{I}})$	e.o.c. ($\vert\!\vert\!\vert e \vert\!\vert\!\vert_{s,h}$)
adaptive refinement ($\sigma = 0.4$)			
majorant with $\boldsymbol{y}_h := \mathrm{argmin}_{\boldsymbol{y}_h \in Y_h} \overline{\mathrm{M}}$			
2	2.5516e-03	1.07	3.42
4	2.2734e-04	1.41	2.36
6	2.9493e-05	1.08	2.70
8	4.8121e-06	1.12	1.56
majorant with $\boldsymbol{y}_h = \nabla_x u_h$ (implies residual-type estimate)			
2	2.5516e-03	4.52	3.42
4	2.2734e-04	1.80	1.49
6	2.6218e-05	3.57	2.43
8	3.1014e-06	3.43	2.74
majorant with equilibrated fluxes (implies equilibration-type estimate)			
2	2.5516e-03	30.14	3.42
4	2.1893e-04	705.39	2.10
6	3.7533e-05	3663.95	2.78
8	1.0382e-05	33422.45	2.11

Efficiency of $\overline{\mathrm{M}}^{\mathrm{I}}$ w.r.t. three approaches of the $\boldsymbol{y}_h \in \oplus^2 S^3_{5h}$ reconstruction.

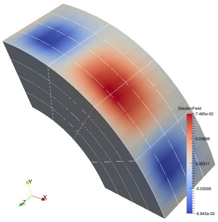
Example 4. Robustness to non-trivial domains

Given data:

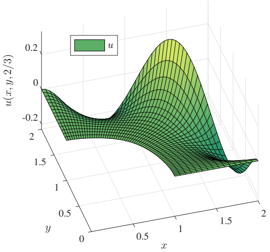
- Ω is a quarter-annulus, $T = 1$
- $u = (1-x)\,x^2\,(1-y)\,y^2\,(1-t)\,t^2$
- $f = \dots$
- $u_D = u$

Discretisation:

- $u_h \in S_h^2$
- $\mathbf{y}_h \in \oplus^2 S_{2h}^3$
- $w_h \in S_{2h}^3$



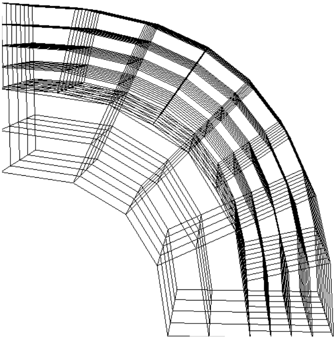
(a)



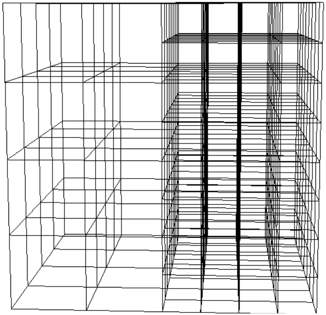
(b)

(a) u in 2d+t axis. (b) u at $t = \frac{2}{3}$.

Example 4. Meshes on parametric and physical domains for $\mathbb{M}_{\mathrm{BULK}}(0.6)$

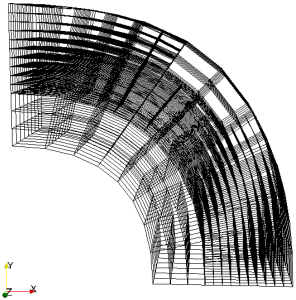


ref. 1: $\hat{Q}$ and $\hat{\mathcal{K}}_h$

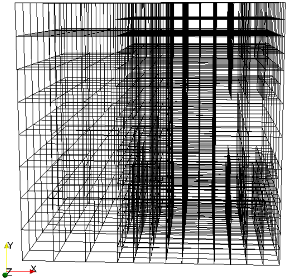


ref. 1: Q and $\mathcal{K}_h$

Example 4. Meshes on parametric and physical domains for $\mathbb{M}_{\mathrm{BULK}}(0.6)$

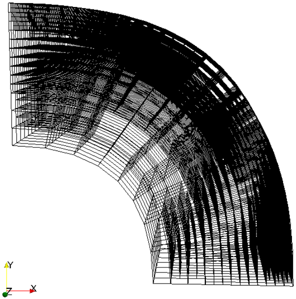


ref. 2: $\hat{Q}$ and $\hat{\mathcal{K}}_h$

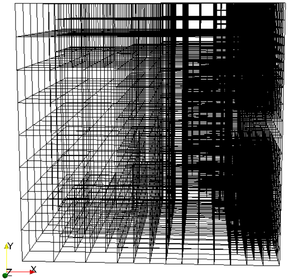


ref. 2: Q and $\mathcal{K}_h$

Example 4. Meshes on parametric and physical domains for $\mathbb{M}_{\mathrm{BULK}}(0.6)$



ref. 3: $\hat{Q}$ and $\hat{\mathcal{K}}_h$

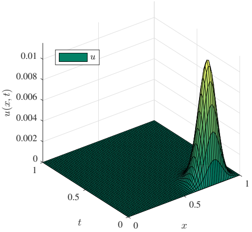


ref. 3: Q and $\mathcal{K}_h$

Example 3. Robustness to solutions with sharp local Gaussian jumps

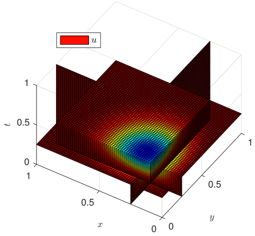
Example 3-1: 1d + t

- $\Omega = (0,1),\ T = 1$
- $u = (x^2 - x)\,(t^2 - t)\, e^{-100\,|(x,t)-(0.8,0.05)|}$
- $f = \ldots$
- $u_D = 0$



Example 3-2: 2d + t

- $\Omega = (0,1)^2,\ T = 2$
- $u = (x^2 - x)\,(y^2 - y)\,(t^2 - t)\, e^{-100\,|(x,y,t)-(0.25,0.25,0.25)|}$
- $f = \ldots$
- $u_D = 0$

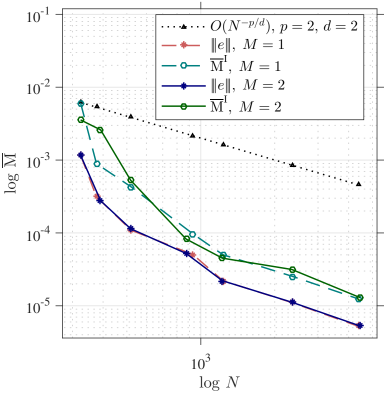


Example 3. Adaptive refinement, $u_h \in S_h^2$

# ref.	$\|\!\|\!\| e \|\!\|\!\|_Q$	$I_{\rm eff}(\overline{\rm M}^{\rm I})$	$I_{\rm eff}(\overline{\rm M}^{\rm II})$	$\|\!\|\!\| e \|\!\|\!\|_{loc,h}$	$\|\!\|\!\| e \|\!\|\!\|_{\mathcal{L}}$	$I_{\rm eff}(\mathbb{E}\mathrm{d})$	e.o.c. ($\|\!\|\!\| e \|\!\|\!\|_{loc,h}$)	e.o.c. ($\|\!\|\!\| e \|\!\|\!\|_{\mathcal{L}}$)
(a) $u_h \in S_h^2$, $\mathbf{y}_h \in \oplus^2 S_h^3$, and $w_h \in S_h^3$								
2	3.1311e-04	2.85	1.55	3.1335e-04	5.6510e-02	1.00	17.71	8.64
3	1.0915e-04	3.93	1.73	1.0944e-04	3.1506e-02	1.00	6.49	3.60
5	2.2033e-05	2.27	1.36	2.2042e-05	1.4796e-02	1.00	5.87	3.59
7	5.2517e-06	2.38	1.22	5.2526e-06	7.2473e-03	1.00	2.41	1.27
(b) $u_h \in S_h^2$, $\mathbf{y}_h \in \oplus^2 S_{2h}^6$, and $w_h \in S_{2h}^6$								
2	2.7623e-04	9.39	2.38	2.7647e-04	5.4452e-02	1.00	15.35	7.24
3	1.1419e-04	4.62	1.79	1.1446e-04	3.1695e-02	1.00	6.27	3.85
5	2.2089e-05	2.04	1.13	2.2099e-05	1.4911e-02	1.00	5.07	3.04
7	5.2825e-06	2.45	1.24	5.2837e-06	7.1577e-03	1.00	2.35	1.24

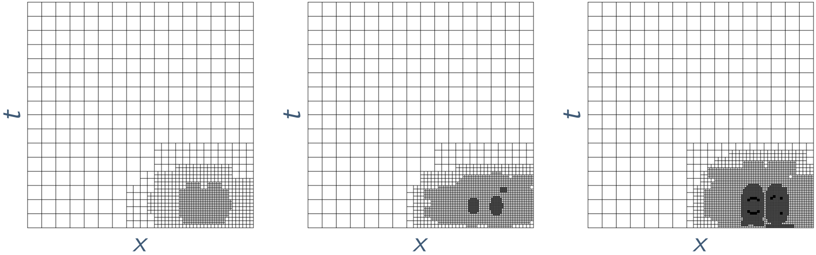
Efficiency of $\overline{\rm M}^{\rm I}$, $\overline{\rm M}^{\rm II}$, $\overline{\rm M}^{\rm I}_{s,h}$, and $\mathbb{E}\mathrm{d}$.

Example 3. Error order of convergence for $u_h \in S_h^2$



The majorant and e.o.c. for (a) $\boldsymbol{y}_h \in \oplus^2 S_h^3$, and $w_h \in S_h^3$ and (b) $\boldsymbol{y}_h \in \oplus^2 S_{2h}^6$, and $w_h \in S_{2h}^6$.

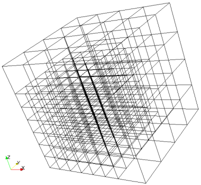
Example 3-1. Mesh refinement for $\mathbb{M}_{\mathrm{BULK}}(0.6)$



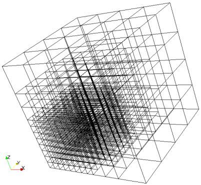
ref. 4 ref. 5 ref. 6

Meshes obtained on the refinement steps 4–6.

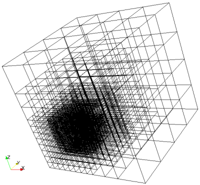
Example 3-2. Mesh refinement for $\mathbb{M}_{\rm BULK}(0.6)$



ref. 1: Q and $\mathcal{K}_h$



ref. 2: Q and $\mathcal{K}_h$



ref. 3: Q and $\mathcal{K}_h$

Meshes obtained on the refinement steps 1–3 $u_h \in S_h^2$, $\boldsymbol{y}_h \in \oplus^2 S_h^3$ and $w_h \in S_h^3$