Some Electromagnetic Wave Propagation Models for Moving Media

AANMPDE11

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Introduction

Maxwell's equations (Maxwell 1861, Gibbs, Heaviside, Hertz 1881).

$$\begin{aligned} \partial_t \mu H + \operatorname{curl} E &= K, \\ \partial_t \varepsilon E + \sigma E - \operatorname{curl} H &= -J, \end{aligned} \tag{1}$$

Compact form nowadays standard 6-vector form (Minkowski 1908, G. Schmidt 1967, R. Leis 1968):

$$M_0 = \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix}, M_1 = \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix}, A_{\mathrm{Max}} = \begin{pmatrix} 0 & -\operatorname{curl} \\ \mathring{\operatorname{curl}} & 0 \end{pmatrix},$$

$$(\partial_t M_0 + M_1 + A_{\mathrm{Max}}) \begin{pmatrix} E \\ H \end{pmatrix} = \begin{pmatrix} -J \\ K \end{pmatrix}.$$

Introduction

Observer in motion (with velocity field v), which leads to a Maxwell system with drift term

$$\begin{aligned} \partial_t \mu H + \tfrac{\partial}{\partial v} \mu H + \operatorname{curl} E &= K, \\ \partial_t \varepsilon E + \tfrac{\partial}{\partial v} \varepsilon E - \operatorname{curl} H &= -J, \end{aligned} \tag{2}$$

sometimes referred to as *Maxwell-Hertz-Cohn system* (Cohn 1901, Hertz 1908). We shall discuss this and related systems in a unified functional-analytical framework (*Evo-Systems*), which facilitates comparison.

$$\left(\partial_t M_0 + M_1 + A_{\mathrm{Max}} + \frac{\partial}{\partial v} M_0\right) \begin{pmatrix} E \\ H \end{pmatrix} = \begin{pmatrix} -J \\ K \end{pmatrix}$$

with $M_0 = \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix}$ $M_1 = \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix}$ as before.

The Maxwell-Hertz-Cohn model has been superseded by a relativistic approach: Maxwell-Minkowski model, Minkowski 1908. Survey: Sommerfeld, Electrodynamics Lectures Vol. 3, 1952

Preliminaries

The Linear Solution Theory.

Solve the *linear* equation:

$$Au = f,$$

$A : D(A) \subseteq H \to H$, H a complete, real inner product space, i.e. a Hilbert space with inner product $\langle \cdot | \cdot \rangle_H$ and norm $|\cdot|_H$.

Adjoint operator $A^* : D(A^*) \subseteq H \to H$:

$$\langle Ax|y\rangle_H = \langle x|A^* y\rangle_H, x \in D(A), y \in D(A^*).$$

By the projection theorem we have with

$$R(A) := \{y|y = Ax, x \in D(A)\}, N(A) := \{x|Ax = 0\},$$

$$H = \overline{R(A)} \oplus R(A)^\perp = \overline{R(A)} \oplus N(A^*) \qquad H = \overline{R(A^*)} \oplus R(A^*)^\perp = \overline{R(A^*)} \oplus N(A).$$

For every $f \in H$ there is a unique $u \in D(A)$ such that

$$Au = f$$

if and only if (Hadamard's requirements)

- $N(A) = \{0\}$ (uniqueness),
- $N(A^*) = \{0\}$ (approximate solvability),
- $R(A) = \overline{R(A)}$ (continuous dependence on data).

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Good Case: "Positivity".

If A and A^*, with $A = A^{**}$, are strictly positive definite, i.e. for some positive c_0

$$|x|_H \, |Ax|_H \geq \langle x|Ax\rangle_H \geq c_0 \, \langle x|x\rangle_H = c_0 \, |x|_H^2 \,,$$
$$|y|_H \, |A^* y|_H \geq \langle y|A^* y\rangle_H \geq c_0 \, \langle y|y\rangle_H = c_0 \, |y|_H^2 \,,$$

then we have unique existence of solution u for every $f \in H$, in other words $u = A^{-1} f$.

Moreover,

$$|u|_H = \left|A^{-1} f\right|_H \leq \frac{1}{c_0} \, |f|_H \ \text{ or } \ \left\|A^{-1}\right\| \leq \frac{1}{c_0}$$

(continuous dependence on the data; Hadamard's requirements for well-posedness).

Three Key Ideas for Evo-Systems

The framework in which we discuss dynamic problems rests on the previous discussion and the following three key concepts:

1. The time derivative operator ∂_t is a *strictly positive definite* linear operator.
2. Evo-systems are *strictly positive definite* (yielding Hadamard well-posedness!).
3. Evo-systems are – beyond Hadamard well-posedness – characterized by *causality* (*!*).

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What are Evo-Systems?

General Form (π 2009):

$$\partial_t V + AU = f \text{ on }]0,\infty[,$$
$$V(0+) = \Phi,$$

where A is skew-selfadjoint, i.e. $A = -A^*$, in which case $\langle x|Ax\rangle_H = 0$ for $x \in D(A)$, in an underlying Hilbert space H.

Without much loss of generality: $\Phi = 0$.

Thus

$$\partial_t V + AU = f \text{ on } \mathbb{R}. \tag{3}$$

Material Law:

$$V = \mathscr{M} U. \tag{4}$$

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The Shape of Evo-Systems

General Form of Evolutionary Problems:

$$\partial_t V + AU = f \text{ on } \mathbb{R},\ V = \mathscr{M}U.$$

Evo-Systems:

$$(\partial_t \mathscr{M} + A)\, U = f.$$

Solution Theory: Does the operator

$$(\partial_t \mathscr{M} + A)^{-1}$$

exist as a continuous linear mapping on a suitable real Hilbert space?

Which "*suitable*" real Hilbert space?

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The Time Derivative as a Normal Operator

Exponential weight function $t \mapsto \exp(-\rho\, t)$, $\rho \in \mathbb{R}$, generates a weighted L^2-space $H_{\rho,0}(\mathbb{R}, H)$ (inner product $\langle \cdot | \cdot \rangle_{\rho,0,0}$, norm: $|\cdot|_{\rho,0,0}$)

$$(\varphi, \psi) \mapsto \int_{\mathbb{R}} \langle \varphi(t) | \psi(t) \rangle_H \exp(-2\rho\, t)\, dt.$$

Time-differentiation ∂_t as a closed operator in $H_{\rho,0}(\mathbb{R}, H)$ induced by

$$\begin{aligned} \mathring{C}_1(\mathbb{R}, H) \subseteq H_{\rho,0}(\mathbb{R}, H) &\to H_{\rho,0}(\mathbb{R}, H), \\ \varphi &\mapsto \varphi'. \end{aligned}$$

The Time Derivative as a Normal Operator

Time-differentiation ∂_t is a *normal* operator in $H_{\rho,0}(\mathbb{R}, H)$

$$\partial_t = \mathfrak{sym}(\partial_t) + \mathfrak{skew}(\partial_t) = \frac{1}{2}(\partial_t + \partial_t^*) + \frac{1}{2}(\partial_t - \partial_t^*)$$

with $\mathfrak{sym}(\partial_t)$ self-adjoint and $\mathfrak{skew}(\partial_t)$ skew-selfadjoint with commuting resolvents:

$$\mathfrak{sym}(\partial_t) = \rho.$$

For $\rho \in \mathbb{R} \setminus \{0\}$: continuous invertibility of ∂_t.

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Basics of the Solution Theory $H_{\rho,0}(\mathbb{R},H)$

Simple Evo-Systems: $\mathscr{M} = M\left(\partial_t^{-1}\right) = M_0 + \partial_t^{-1} M_1$

$$(\partial_t M_0 + M_1 + A)\, U = F$$

Normal Form: When is $(\partial_t M_0 + M_1 + A)$ (and its adjoint) strictly positive definite in $H_{\rho,0}(\mathbb{R},H)$ (for all sufficiently large $\rho \in\,]0,\infty[$)?

Assumptions (E):

- A skew-selfadjoint in H (lifted to $H_{\rho,0}(\mathbb{R},H)$),
- M_0 **selfadjoint**[1] and $\langle u|(\rho M_0 + \mathfrak{sym}(M_1))\,u\rangle_H \geq c_0 \langle u|u\rangle_H$ for all $u \in H$ and all sufficiently large $\rho \in\,]0,\infty[$.

The latter is for example the case if

- M_0 **selfadjoint, strictly positive definite on its range,**
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[1] $M_0 = \mathfrak{sym}(M_0)$

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Assumptions (E) imply

$$\left\langle \chi_{]-\infty,a[}(\mathbf{m}_0)\, U \middle| \left(\partial_t M_0 + \mathfrak{sym}(M_1)\right) U \right\rangle_{\rho,0,0} =$$
$$= \left\langle \chi_{]-\infty,a[}(\mathbf{m}_0)\, U \middle| \chi_{]-\infty,a[}(\mathbf{m}_0)\left(\partial_t M_0 + M_1 + A\right) U \right\rangle_{\rho,0,0}$$
$$\geq c_0 \left| \chi_{]-\infty,a[}(\mathbf{m}_0)\, U \right|^2_{\rho,0,0}$$

and so also ($M^* = (\mathfrak{sym}(M) + \mathfrak{skew}(M))^* = \mathfrak{sym}(M) - \mathfrak{skew}(M)$)

$$\langle U | \left(\rho M_0 + \mathfrak{sym}(M_1)\right) U \rangle_{\rho,0,0} =$$
$$= \left\langle U \middle| \left(\partial_t M_0 + M_1 + A\right)^* U \right\rangle_{\rho,0,0} \geq c_0 \left| U \right|^2_{\rho,0,0}$$

uniformly for $U \in D(\partial_t) \cap D(A)$ and all $a \in \mathbb{R}$, $\rho \in]\rho_0, \infty[$, where ρ_0 is sufficiently large).

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$$\begin{aligned}&\left\langle \chi_{]-\infty,a[}(\mathbf{m}_0)\,U\middle|\left(\partial_t M_0+\mathfrak{sym}\left(M_1\right)\right)U\right\rangle_{\rho,0,0}=\\&=\left\langle \chi_{]-\infty,a[}(\mathbf{m}_0)\,U\middle|\chi_{]-\infty,a[}(\mathbf{m}_0)\left(\partial_t M_0+M_1+A\right)U\right\rangle_{\rho,0,0}\\&\geq c_0\left|\chi_{]-\infty,a[}(\mathbf{m}_0)\,U\right|_{\rho,0,0}^2\end{aligned}$$

and so also $(M^*=(\mathfrak{sym}(M)+\mathfrak{skew}(M))^*=\mathfrak{sym}(M)-\mathfrak{skew}(M))$

$$\begin{aligned}&\left\langle U\middle|\left(\rho M_0+\mathfrak{sym}\left(M_1\right)\right)U\right\rangle_{\rho,0,0}=\\&=\left\langle U\middle|\left(\partial_t M_0+M_1+A\right)^*U\right\rangle_{\rho,0,0}\geq c_0\left|U\right|_{\rho,0,0}^2\end{aligned}$$

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The Solution Theorem

Theorem

Let M_0, M_1 and A satisfy ***Assumptions (E)****. Then we have for all sufficiently large $\rho \in]0,\infty[$ that for every $f \in H_{\rho,0}(\mathbb{R}, H)$ there is a unique solution $U \in H_{\rho,0}(\mathbb{R}, H)$ of the problem*

$$(\partial_t M_0 + M_1 + A)\, U = f\,.$$

The solution operator $(\partial_t M_0 + M_1 + A)^{-1}$ is continuous and causal on $H_{\rho,0}(\mathbb{R}, H)$.

The Maxwell-Gibbs-Heaviside Equations

Three-dimensional Maxwell equations ($G = \mathring{\text{curl}}, \quad G^* = \text{curl}$)

$$\partial_t \begin{pmatrix} D \\ B \end{pmatrix} + A_{\text{Max}} \begin{pmatrix} E \\ H \end{pmatrix} = \mathscr{J},$$

with

$$A_{\text{Max}} := \begin{pmatrix} 0 & -\text{curl} \\ \mathring{\text{curl}} & 0 \end{pmatrix}$$

material law

$$\begin{pmatrix} D \\ B \end{pmatrix} = \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} + \partial_t^{-1} \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix},$$

where $\rho\varepsilon + \mathfrak{sym}\sigma$, μ selfadjoint, strictly positive definite (for all sufficiently large $\rho \in]0,\infty[$).

The Maxwell-Hertz-Cohn Model

By vector analysis calculations we may re-write Maxwell's equations with a drift term in normal form as

$$\left(\partial_0 M_0 + A_{\mathrm{Max}}\right)\begin{pmatrix}\widetilde{E}\\ \widetilde{H}\end{pmatrix} = \begin{pmatrix}-J\\ K\end{pmatrix}, \tag{5}$$

with

$$\begin{pmatrix}\widetilde{E}\\ \widetilde{H}\end{pmatrix} = \begin{pmatrix}1 & -v\times\mu\\ v\times\varepsilon & 1\end{pmatrix}\begin{pmatrix}E\\ H\end{pmatrix},$$

as new unknowns and

$$M_0 := \begin{pmatrix}\varepsilon & 0\\ 0 & \mu\end{pmatrix}^{1/2}\begin{pmatrix}1 & -\frac{v}{c}\times\\ \frac{v}{c}\times & 1\end{pmatrix}^{-1}\begin{pmatrix}\varepsilon & 0\\ 0 & \mu\end{pmatrix}^{1/2} = \begin{pmatrix}\varepsilon & \varepsilon v\times\mu\\ -\mu v\times\varepsilon & \mu\end{pmatrix} + \cdots$$

which is strictly positive definite if and only if

$$\frac{|v|}{c} \leq d_0 < 1.$$

The Maxwell-Minkowski Model

Minkowski derived the constitutive laws of the electromagnetic field in moving media via a Lorentz transformation approach and so has the speed of light as a threshold built in. In normal form

$$(\partial_0 M_0 + A_{\mathrm{Max}})\, U = F$$

$$M_0 =$$

$$= \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix}^{1/2} \left(1 - \beta \begin{pmatrix} (1-\alpha)(1-P_v) & -\frac{v}{|v|}\times \\ \frac{v}{|v|}\times & (1-\alpha)(1-P_v) \end{pmatrix}\right) \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix}^{1/2},$$

where $P_v x = \left\langle \frac{v}{|v|} \middle| x \right\rangle \frac{v}{|v|}$.

The Maxwell-Minkowski Model

Condition for positive definiteness

$$\frac{1-\frac{|v|^2}{c_0^2}}{1-\frac{|v|^2}{c^2}} =: \alpha > \beta := \left(N - N^{-1}\right)\frac{\frac{|v|}{c_0}}{1-\frac{|v|^2}{c^2}}$$

with the refractive index $N := \frac{c_0}{c}$. This yields

$$\frac{|v|}{c} < 1 \leq N.$$

We can now assume variable coefficients again, if we assume that ε, μ are scalar multipliers (isotropic media).

Panofsky & Phillips 1962, Epstein 1963, Tai 1964, van Bladel 1973-1984, Cooper & Strauss 1985, Georgiev 1989, Ivezic 2001-today, Rousseau 2006-today, Ferencz 2011

'instantaneous rest-frame hypothesis' (van Bladel) .

Same 'magic trick' used earlier in the Maxwell-Hertz-Cohn model!

The Maxwell-Hertz-Cohn Model – revisited

A more subtle approach avoids the speed constraint:

If we do not transform the drift term, we are led to a different perspective of electromagnetic fields with drift, indeed without resorting to 'magic tricks'. For this, however, we restrict $\pm v$ to have a *fixed direction.* in order to avoid more involved considerations. By appropriate rotation of the Euclidean coordinates, we may then indeed assume without loss of generality that

$$v = \alpha e_3 := \alpha \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Main tool: the following elementary abstract lemma on operator sums.

Lemma

Let A, B be closed densely defined linear operators in a Hilbert space X such that $D(A+B) = D(A) \cap D(B)$ is dense in X. Assume there is a family of continuous linear operators $(C_\eta)_{\eta \in]0,1[}$ such that $C_\eta [D(A)] \subseteq D(A)$, $C_\eta [D(B)] \subseteq D(B)$, $C_\eta^* \overset{s}{\to} 1, [A, C_\eta]^* \overset{s}{\to} 0, [B, C_\eta]^* \overset{s}{\to} 0$ as η tends to zero. Moreover, let AC_η be a continuous linear operator in X. Then

$$\overline{A^* + B^*} = (A+B)^* .$$

We make the general choice that $\Omega \subseteq \mathbb{R}^3$ is a non-empty open set such that

$$[\Omega] + [[\mathbb{R}]\, e_3] \subseteq \Omega,$$

i.e. Ω is a cylindrical domain $\Omega = Q \times \mathbb{R}$ with cross section Q. In this situation, it is an easy exercise to show that ∂_3 is skew-selfadjoint in $L^2(\Omega)$. The role of C_η will here be played by resolvent terms of the form $(1 \pm \eta\partial_3)^{-1}$.

For the case that motivated our considerations we record the following result.

Theorem

Let $\alpha, \partial_3\alpha \in L^\infty(\mathbb{R}\times\Omega,\mathbb{R})$ *and* ε,μ *selfadjoint, strictly positive definite, commuting with* ∂_3 *in* $L^2(\Omega,\mathbb{R}^3)$*. Then*

$$\overline{\left(\rho\begin{pmatrix}\varepsilon & 0\\ 0 & \mu\end{pmatrix}+\begin{pmatrix}\sigma & 0\\ 0 & 0\end{pmatrix}+\mathrm{sym}\left(\frac{\partial}{\partial v}\begin{pmatrix}\varepsilon & 0\\ 0 & \mu\end{pmatrix}\right)+\right.}\overline{\left.+(\partial_t-\rho)\begin{pmatrix}\varepsilon & 0\\ 0 & \mu\end{pmatrix}+\mathrm{skew}\left(\frac{\partial}{\partial v}\begin{pmatrix}\varepsilon & 0\\ 0 & \mu\end{pmatrix}\right)+\begin{pmatrix}0 & -\operatorname{curl}\\ \mathring{\operatorname{curl}} & 0\end{pmatrix}\right)}\begin{pmatrix}E\\ H\end{pmatrix}=\begin{pmatrix}-J\\ K\end{pmatrix}$$

where

$$v=\begin{pmatrix}0\\ 0\\ \alpha\end{pmatrix},$$

describes a class of well-posed problems with a causal solution operator if $\rho\in]0,\infty[$ *is chosen sufficiently large.*

THE END