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Adaptive boundary element methods

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Introduction to BEM

Fundamental solution & Newton potential

- Newton kernel $G(z) := \begin{cases} -\frac{1}{2\pi} \log|z| & \text{for } d = 2 \\ +\frac{1}{4\pi} \frac{1}{|z|} & \text{for } d = 3 \end{cases}$
- note $|S_2^2| = 2\pi$ and $|S_2^3| = 4\pi$ for unit sphere $S_2^d \subset \mathbb{R}^d$
- $f \in L^2(\Omega)$
- $\widetilde{N} f(x) := \int_\Omega G(x-y) f(y)\, dy$ Newton potential

$\Longrightarrow$ $f = -\Delta(\widetilde{N} f)$

Representation formula

Proposition (representation formula for interior problem)

- $\Omega \subset \mathbb{R}^d$ bounded
- $u \in C^2(\overline{\Omega}), \quad f := -\Delta u, \quad \phi := \partial_n u$

$$\Longrightarrow \quad u(x) = \int_\Omega G(x-y) f(y)\, dy + \int_\Gamma G(x-y)\, \phi(y)\, dy - \int_\Gamma \partial_{n(y)} G(x-y)\, u(y)\, dy$$

$$= \widetilde{N}(x) + \widetilde{V}(x) - \widetilde{K}(x) \qquad \text{for all } x \in \Omega$$

- if trace $u|_\Gamma$ and normal derivative $\partial_n u$ known on Γ

  $\Longrightarrow$ solution of $-\Delta u = f$ can be computed in Ω

Weakly-singular integral equation

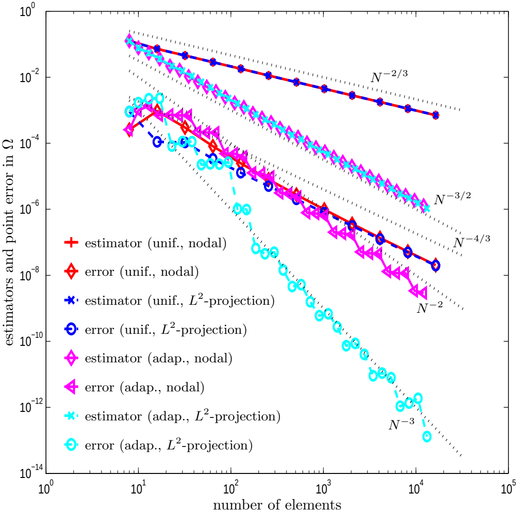
- solve $\Delta u = 0$ in Ω with $u = g$ on Γ
- representation formula $u = \widetilde{V}(\partial_n u) - \widetilde{K} g$ in Ω
- trace $g = V(\partial_n u) - (K - 1/2)g$ on Γ

$$\Longrightarrow \quad \boxed{\begin{aligned} -\Delta u &= 0 \quad \text{in } \Omega \\ u &= g \quad \text{on } \Gamma \end{aligned}} \quad \Longleftrightarrow \quad \boxed{V\phi = (K + 1/2)g \quad \text{on } \Gamma}$$

- has unique solution $\phi = \partial_n u$
- discretization of weakly-singular IE $\Longrightarrow$ approximation $\phi \approx \Phi_\ell$

$\Longrightarrow$ approximation $u \approx U_\ell := \widetilde{V}\Phi_\ell - \widetilde{K} g$

High-order convergence of point errors



Aurada, Ferraz-Leite, Praetorius et al.: Numer. Algorithms 67 (2014)

http://www.asc.tuwien.ac.at/abem/hilbert/

Comments

- ⊖ BEM requires fundamental solution
- ⊖ essentially requires homogeneous forces, i.e., $-\Delta u = 0$
- ⊖ BEM leads to dense matrices

- ⊕ BEM allows for higher convergence rates
- ⊕ BEM only requires surface discretization
- ⊕ BEM can treat unbounded domains

Steinbach: Teubner, 2003 (German), 2008 (English)

Sauter, Schwab: Teubner, 2004 (German), Springer 2011 (English)

Need for adaptivity

Model problem 2D

- weakly-singular integral equation

$$V\phi(x) := -\frac{1}{2\pi}\int_\Gamma \log|x-y|\,\phi(y)\,dy = f(x) \quad \text{for } x\in\Gamma$$

- $\Gamma \subseteq \partial\Omega$ with $\Omega \subset \mathbb{R}^2$ bounded and Lipschitz

Variational formulation

- find solution $\phi \in \widetilde{H}^{-1/2}(\Gamma)$ of

$$\langle\!\langle \phi\,,\psi \rangle\!\rangle := \langle V\phi\,,\psi\rangle_{L^2(\Gamma)} = \langle f\,,\psi\rangle_{L^2(\Gamma)} \quad \text{for all } \psi \in \widetilde{H}^{-1/2}(\Gamma)$$

- $\langle\!\langle \cdot\,,\cdot \rangle\!\rangle$ is scalar product
- induced norm $|\!|\!|\psi|\!|\!| := \langle\!\langle \psi\,,\psi \rangle\!\rangle^{1/2} \simeq \|\psi\|_{\widetilde{H}^{-1/2}(\Gamma)}$
- Riesz theorem $\Longrightarrow$ unique solution ϕ
- ϕ in general *not* computable

Galerkin BEM 2D

- $\mathcal{T}_\ell$ partition of Γ into affine line segments
- $\mathcal{P}^p(\mathcal{T}_\ell)$ space of $\mathcal{T}_\ell$-piecewise polynomials of degree $\le p$

Galerkin formulation

- find solution $\Phi_\ell \in \mathcal{P}^p(\mathcal{T}_\ell)$ of

$$\langle\!\langle \Phi_\ell \,, \Psi_\ell \rangle\!\rangle = \langle f \,, \Psi_\ell \rangle_{L^2(\Gamma)} \quad \text{for all } \Psi_\ell \in \mathcal{P}^p(\mathcal{T}_\ell)$$

- Riesz theorem $\implies$ unique solution Φ_ℓ
- Φ_ℓ computable by solving a linear SPD system
- Céa lemma $\quad |\!|\!| \phi - \Phi_\ell |\!|\!| = \min_{\Psi_\ell \in \mathcal{P}^p(\mathcal{T}_\ell)} |\!|\!| \phi - \Psi_\ell |\!|\!|$

A priori error estimate

- suppose regularity $\phi \in \widetilde{H}^{-1/2}(\Gamma) \cap H^t_{\mathrm{pw}}(\Gamma)$
- $\Phi_\ell \in \mathcal{P}^p(\mathcal{T}_\ell)$ Galerkin solution

$$\Longrightarrow \quad \|\phi - \Phi_\ell\|_{\widetilde{H}^{-1/2}(\Gamma)} \lesssim \|\phi\|_{H^t_{\mathrm{pw}}(\Gamma)} \max_{T \in \mathcal{T}_\ell} \mathrm{diam}(T)^{\min\{p+1,t\}+1/2}$$

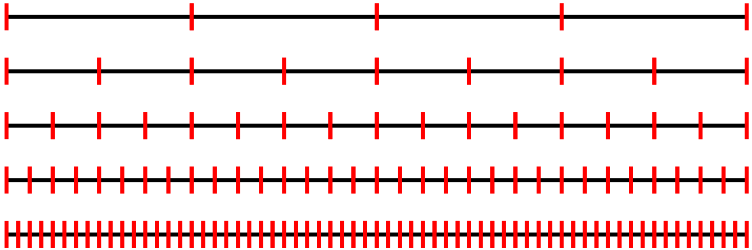
Optimal convergence behavior (for smooth ϕ)

- suppose $h \simeq \mathrm{diam}(T)$ for all $T \in \mathcal{T}_\ell$
- $N := \#\mathcal{T}_\ell \simeq h^{-1}$ (because of 2D)

$$\Longrightarrow \quad |||\phi - \Phi_\ell||| \simeq \|\phi - \Phi_\ell\|_{\widetilde{H}^{-1/2}(\Gamma)} \lesssim h^{p+3/2} \simeq N^{-(p+3/2)}$$

Uniform mesh-refinement 2D

- in each step, all elements $T \in \mathcal{T}_\ell$ are bisected

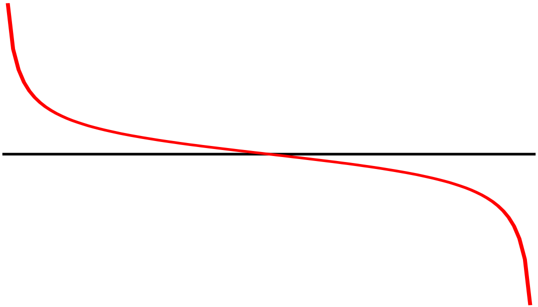


Adaptive mesh-refinement 2D

- in each step, only certain elements are bisected

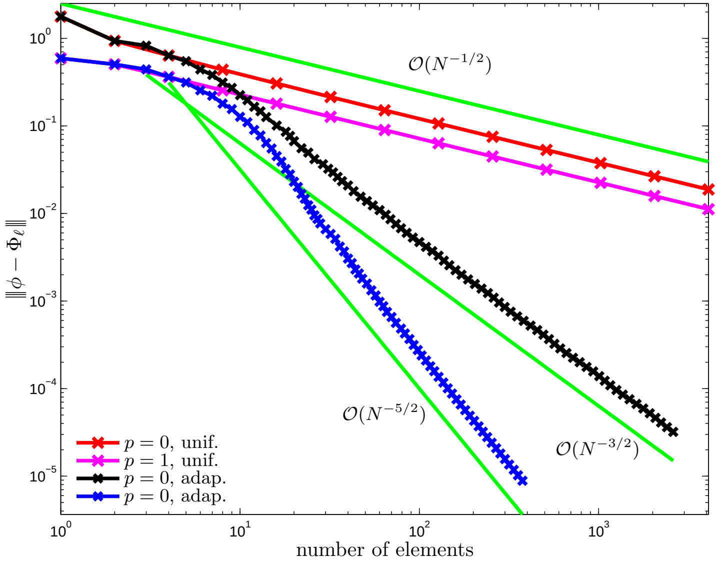
Uniform vs. adaptive mesh-refinement 2D – 1/2

- 1D boundary piece $\Gamma = (-1,1) \times \{0\}$
- $\phi(x,y) = -2x/\sqrt{1-x^2}$ solves $V\phi(x,y) = -x$



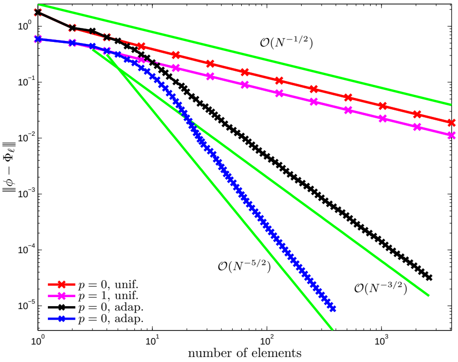
- singularities at $\pm(1,0)$

Uniform vs. adaptive mesh-refinement 2D – 2/2



Observations 2D

- although RHS is smooth, solution exhibits singularities
- uniform refinement can lead to poor convergence rates
- higher p does not help on uniform meshes
- appropriate adaptive refinement recovers optimal rates



Model problem 3D (same as before)

- weakly-singular integral equation

$$V\phi(x) := \frac{1}{4\pi}\int_\Gamma \frac{1}{|x-y|}\,\phi(y)\,dy = f(x) \quad \text{for } x \in \Gamma$$

- $\Gamma \subseteq \partial\Omega$ with $\Omega \subset \mathbb{R}^3$ bounded and Lipschitz

Variational formulation

- find solution $\phi \in \widetilde{H}^{-1/2}(\Gamma)$ of

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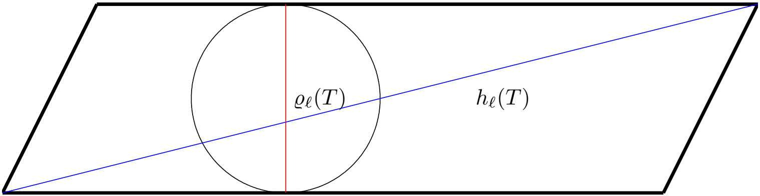
- induced norm $|\!|\!|\psi|\!|\!| := \langle\!\langle \psi\,,\psi \rangle\!\rangle^{1/2} \simeq \|\psi\|_{\widetilde{H}^{-1/2}(\Gamma)}$
- Riesz theorem $\Longrightarrow$ unique solution ϕ
- ϕ in general *not* computable

Galerkin BEM 3D

- as before: discretization $\phi \approx \Phi_\ell \in \mathcal{P}^p(\mathcal{T}_\ell)$
- mesh $\mathcal{T}_\ell$ of Γ



A priori error estimate (same as before)

- suppose regularity $\phi \in \widetilde{H}^{-1/2}(\Gamma) \cap H^t_{\rm pw}(\Gamma)$
- $\Phi_\ell \in \mathcal{P}^p(\mathcal{T}_\ell)$ Galerkin solution

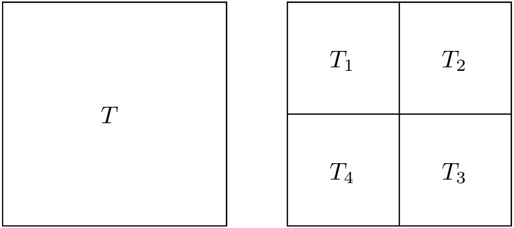
$$\Longrightarrow \quad \|\phi - \Phi_\ell\|_{\widetilde{H}^{-1/2}(\Gamma)} \lesssim \|\phi\|_{H^t_{\rm pw}(\Gamma)} \max_{T\in\mathcal{T}_\ell} \mathrm{diam}(T)^{\min\{p+1,t\}+1/2}$$

Optimal convergence behavior (for smooth ϕ)

- suppose $h \simeq \mathrm{diam}(T) \simeq |T|^{1/2}$ for all $T \in \mathcal{T}_\ell$
- $N := \#\mathcal{T}_\ell \simeq h^{-2}$ (because of 3D)

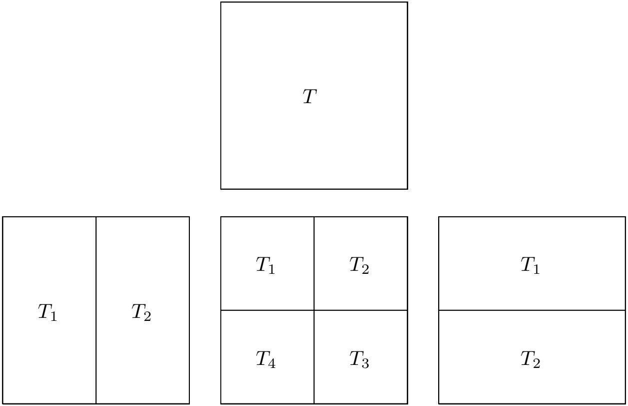
$$\Longrightarrow \quad |||\phi - \Phi_\ell||| \simeq \|\phi - \Phi_\ell\|_{\widetilde{H}^{-1/2}(\Gamma)} \lesssim h^{p+3/2} \simeq N^{-(p+3/2)/2}$$

Isotropic mesh-refinement



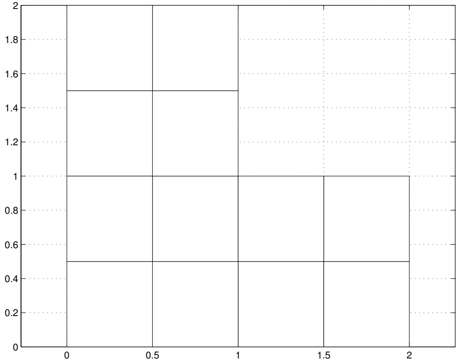
- marked element T is split into $T_1, \dots, T_4$
- either all elements are refined (uniform refinement)
- or only some elements (adaptive isotropic refinement)

Anisotropic mesh-refinement



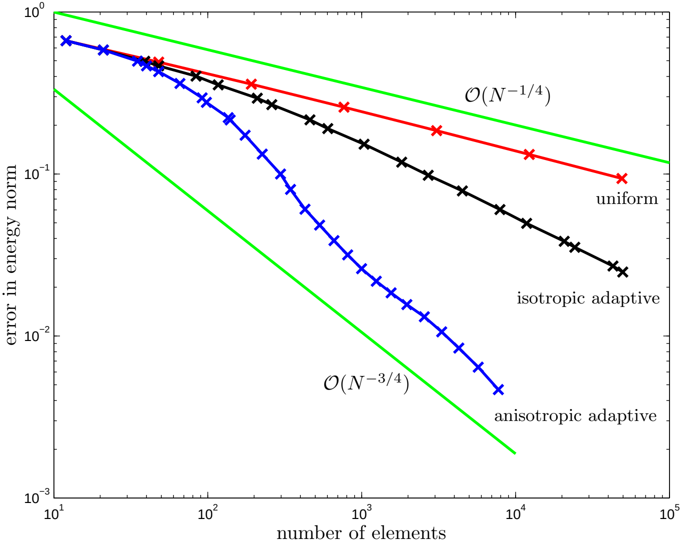
- marked element T can be split anisotropically

Uniform vs. adaptive mesh-refinement 3D – 1/3

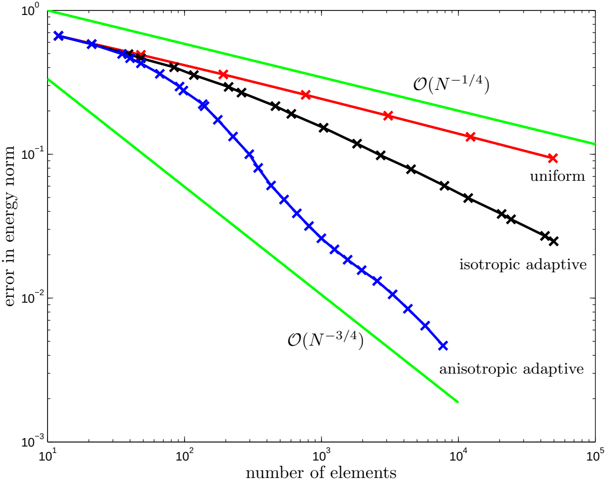


- consider $V\phi = 1$ on L-shaped screen $L \times \{0\} \subset \mathbb{R}^3$
- initial mesh $\mathcal{T}_0$ with $N = 12$ elements
- lowest-order BEM $p = 0$

Uniform vs. adaptive mesh-refinement 3D – 2/3



Uniform vs. adaptive mesh-refinement 3D – 3/3



- error $\simeq N^{-1/t}$

  $\Longrightarrow \quad N \simeq \text{error}^{-t}$

- uniform $t = 4$

  $\Longrightarrow \quad N = 10^8$

  for error $\approx 10^{-2}$

- isotropic $t = 2$

  $\Longrightarrow \quad N = 10^6$

  for error $\approx 10^{-2}$

Higher-order polynomials? Yes, but...

Heuristics (counting argument along edge with singularity)

- p polynomial degree
- $\alpha_{\rm opt}$ optimal rate with anisotropic elements
- $\alpha_{\rm unif}$ reduced rate for uniform mesh-refinement
- $\alpha_{\rm iso}$ convergence rate with isotropic elements

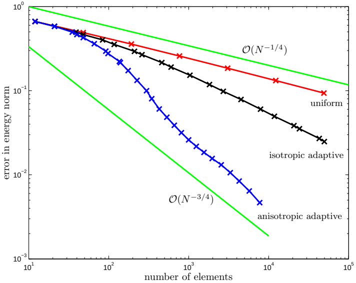
$\Longrightarrow \quad \alpha_{\rm iso} \le \min\{\alpha_{\rm opt}, 2\,\alpha_{\rm unif}\}$

- $\alpha_{\rm opt} = \alpha_{\rm opt}(p)$
- but: $\alpha_{\rm unif}$ essentially independent of p

Gantner: PhD thesis, TU Wien, 2017

Observations 3D (some are new)

- although RHS is smooth, solution ϕ exhibits singularities
  - singularities along edges and at corners
- convergence for uniform refinement can be poor
- new: isotropic adaptive refinement only improves convergence rate



- new: optimal convergence requires anisotropic elements
  - higher-order polynomials usually pay only on anisotropic meshes

Simple h-h/2 estimator

Adaptive algorithm

- initial mesh $\mathcal{T}_0$
- adaptivity parameter $0 < \theta \le 1$
- refinement indicators $\mu_\ell(T) \approx \|\phi - \Phi_\ell\|_T$

- $\theta \ll 1 \implies$ only few marked elts
- $\theta \approx 1 \implies$ essentially uniform refinement

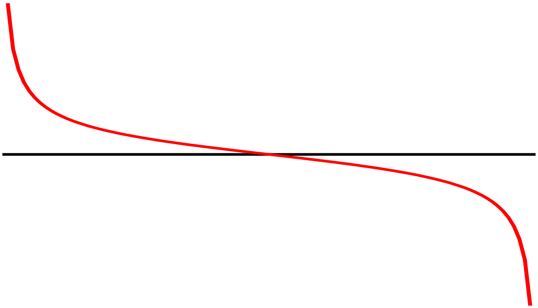
For all $\ell = 0, 1, 2, \ldots,$ iterate

1. compute discrete solution Φ_ℓ for mesh $\mathcal{T}_\ell$, where $\mathcal{X}_\ell = \mathcal{X}_\ell(\mathcal{T}_\ell)$
2. for all $T \in \mathcal{T}_\ell$, compute $\mu_\ell(T)$
3. find (essentially minimal) set $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ s.t.
$$\theta \sum_{T \in \mathcal{T}_\ell} \mu_\ell(T)^2 \le \sum_{T \in \mathcal{M}_\ell} \mu_\ell(T)^2$$
4. refine (at least) marked elements $T \in \mathcal{M}_\ell$ to obtain $\mathcal{T}_{\ell+1}$

Dörfler: SINUM 33 (1996)

h-h/2 Error estimator 1/8

- 1D boundary piece $\Gamma = (-1, 1) \times \{0\}$
- $\phi(x, y) = -2x/\sqrt{1 - x^2}$ solves $V\phi(x, y) = -x$



- singularities at $\pm(1, 0)$

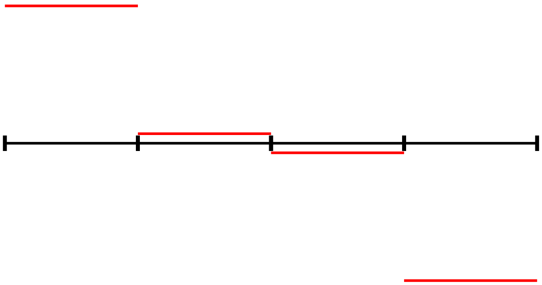
h-h/2 Error estimator 2/8

- start with mesh $\mathcal{T}_\ell$

- refine $\mathcal{T}_\ell$ uniformly to obtain $\widehat{\mathcal{T}}_\ell$

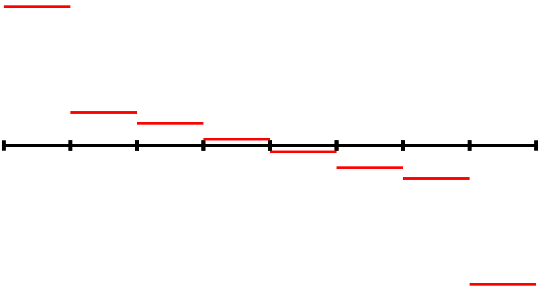
h-h/2 Error estimator 3/8

- compute Galerkin solution $\Phi_\ell \in \mathcal{X}_\ell := \mathcal{P}^0(\mathcal{T}_\ell)$



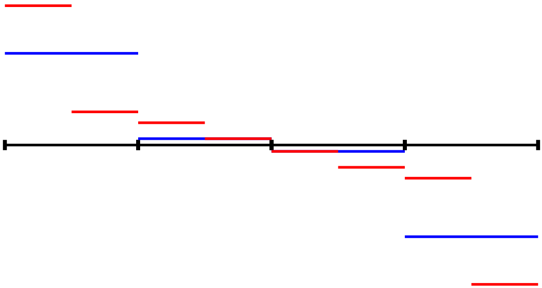
h-h/2 Error estimator 4/8

- compute Galerkin solution $\widehat{\Phi}_\ell \in \widehat{\mathcal{X}}_\ell := \mathcal{P}^0(\widehat{\mathcal{T}}_\ell)$



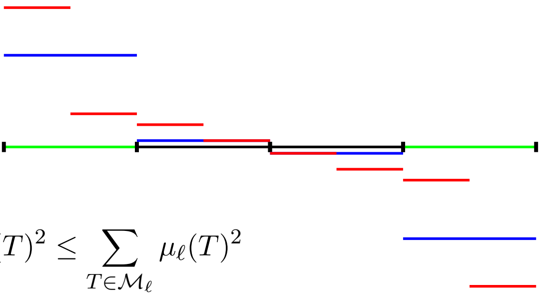
h-h/2 Error estimator 5/8

- compare $\Phi_\ell \in \mathcal{X}_\ell$ and $\widehat{\Phi}_\ell \in \widehat{\mathcal{X}}_\ell$



h-h/2 Error estimator 6/8

- mark $\mathcal{T}_\ell$ locally, where $\mu_\ell(T) = |||\widehat{\Phi}_\ell - \Phi_\ell|||_T$ is large



- $\displaystyle \frac{1}{4}\sum_{T\in\mathcal{T}_\ell}\mu_\ell(T)^2 \le \sum_{T\in\mathcal{M}_\ell}\mu_\ell(T)^2$
- $|||\widehat{\Phi}_\ell - \Phi_\ell|||_T$ not specified yet

h-h/2 Error estimator 7/8

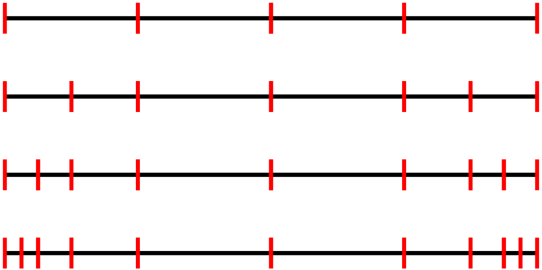
- got marking of old mesh $\mathcal{T}_\ell$

- refine marked elements to obtain new mesh $\mathcal{T}_{\ell+1}$

- refine $\mathcal{T}_{\ell+1}$ uniformly to obtain $\widehat{\mathcal{T}}_{\ell+1}$

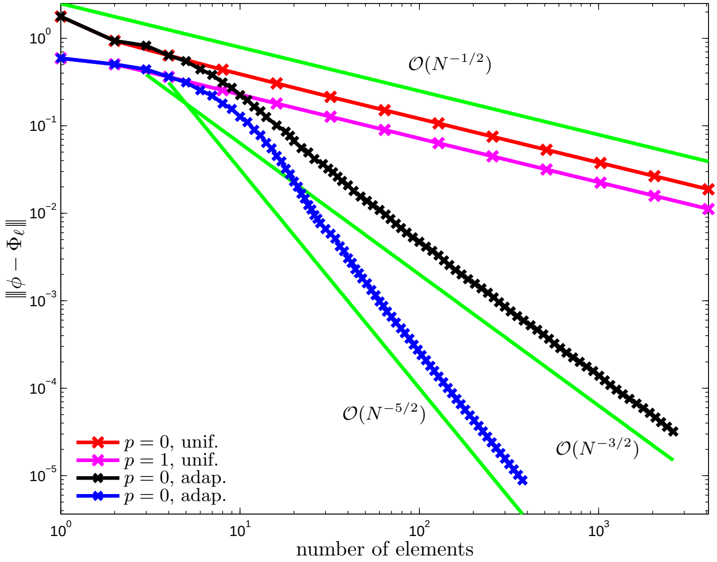
h-h/2 Error estimator 8/8

- creates sequence of meshes $\mathcal{T}_\ell$



- etc.

ABEM results in...



Efficiency & reliability of $\eta_\ell = |||\widehat{\Phi}_\ell - \Phi_\ell|||$

- Pythagoras theorem

$$\Longrightarrow \quad |||\phi - \Phi_\ell|||^2 = |||(\phi - \widehat{\Phi}_\ell) + (\widehat{\Phi}_\ell - \Phi_\ell)|||^2 = |||\phi - \widehat{\Phi}_\ell|||^2 + \eta_\ell^2$$

Efficiency

- $\eta_\ell \le |||\phi - \Phi_\ell|||$

- heuristics: $|||\phi - \widehat{\Phi}_\ell||| \ll |||\phi - \Phi_\ell|||$

Reliability $\Longleftrightarrow$ saturation assumption

- $|||\phi - \Phi_\ell||| \le C_{\rm rel}\, \eta_\ell$ $\qquad C_{\rm rel} = (1 - q_{\rm sat}^2)^{-1/2}$

$\Longleftrightarrow$

- $|||\phi - \widehat{\Phi}_\ell||| \le q_{\rm sat}\, |||\phi - \Phi_\ell|||$ $\qquad$ for some $0 < q_{\rm sat} < 1$

Ferraz-Leite, Praetorius: Computing 83 (2008)

Asymptotic behavior $\Longrightarrow$ saturation assumption

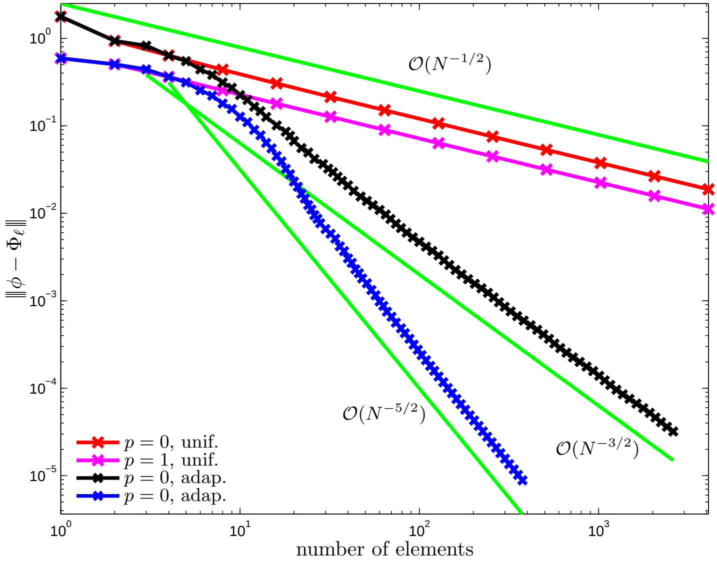
- suppose: asymptotics $|||\phi - \Phi_\ell||| = CN^{-\alpha}$ with $N = \#\mathcal{T}_\ell$
- uniform refinement satisfies $\#\widehat{\mathcal{T}}_\ell = k \cdot \#\mathcal{T}_\ell$ with $k \geq 2$

$\Longrightarrow$ $|||\phi - \widehat{\Phi}_\ell||| = C(k\,\#\mathcal{T}_\ell)^{-\alpha} = k^{-\alpha}\,|||\phi - \Phi_\ell|||$

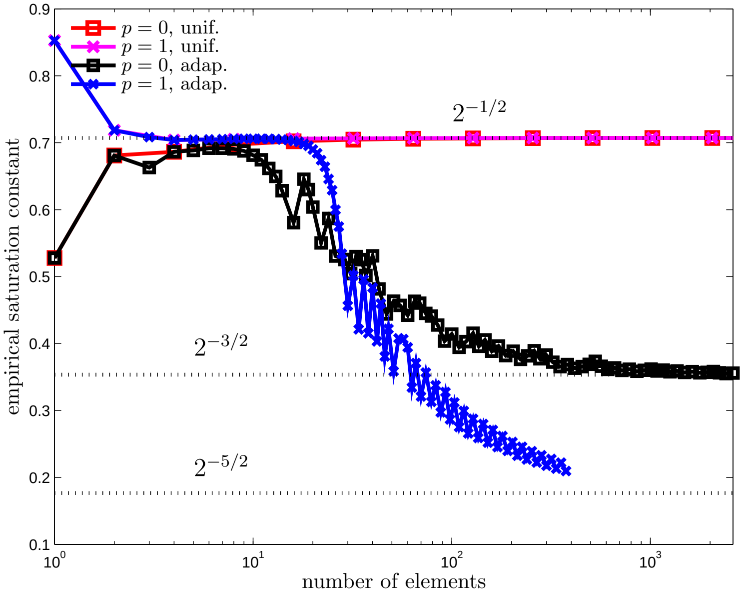
$\Longrightarrow$ $q_{\rm sat} = k^{-\alpha}$

- $k = 2$ for 2D

Saturation assumption for 2D model problem 1/2



Saturation assumption for 2D model problem 2/2



Difficulties & tasks

- $\| \cdot \|_{L^2(\Gamma)}$ is local in the sense of

$$\|\psi\|_{L^2(\Gamma)}^2 = \sum_{T \in \mathcal{T}_\ell} \|\psi\|_{L^2(T)}^2$$

- $|\!|\!| \cdot |\!|\!| \simeq \| \cdot \|_{\widetilde{H}^{-1/2}(\Gamma)}$ is non-local

$$|\!|\!|\psi|\!|\!|^2 = \int_\Gamma \int_\Gamma v(x)\, G(x-y)\, v(y)\, dy\, dx$$

- what is appropriate local contribution $|\!|\!|\widehat{\Phi}_\ell - \Phi_\ell|\!|\!|_T$?
- avoid computation of Φ_ℓ, since $|\!|\!|\phi - \widehat{\Phi}_\ell|\!|\!| \leq |\!|\!|\phi - \Phi_\ell|\!|\!|$

h-h/2 Estimators for BEM

Theorem (Ferraz-Leite, P. '08)

- $\Pi_\ell : L^2(\Gamma) \to \mathcal{P}^p(\mathcal{T}_\ell)$ L^2-orthogonal projection
- $h_\ell, \rho_\ell \in L^\infty(\Gamma)$ local mesh-width functions
- $\eta_\ell = |||\widehat{\Phi}_\ell - \Phi_\ell|||$
- $\widetilde{\eta}_\ell = |||(1-\Pi_\ell)\widehat{\Phi}_\ell|||$
- $\mu_\ell = \|\varrho_\ell^{1/2}(\widehat{\Phi}_\ell - \Phi_\ell)\|_{L^2(\Gamma)}$
- $\widetilde{\mu}_\ell = \|\varrho_\ell^{1/2}(1-\Pi_\ell)\widehat{\Phi}_\ell\|_{L^2(\Gamma)}$ practical h-h/2 estimator

$$\Longrightarrow \quad \widetilde{\mu}_\ell \le \mu_\ell \lesssim \eta_\ell \le \widetilde{\eta}_\ell \lesssim \|h_\ell/\varrho_\ell\|^{1/2}_{L^\infty(\Gamma)}\,\widetilde{\mu}_\ell$$

- proof only based on local inverse / approximation estimates

Ferraz-Leite, Praetorius: Computing 83 (2008)

Remarks

Practical h-h/2 estimator $\widetilde{\mu}_\ell = \|\varrho_\ell^{1/2}(1-\Pi_\ell)\widehat{\Phi}_\ell\|_{L^2(\Gamma)}$

- requires only $\widehat{\Phi}_\ell$, but avoids computation of Φ_ℓ
- local contributions $\mu_\ell(T) := \|\varrho_\ell^{1/2}(1-\Pi_\ell)\widehat{\Phi}_\ell\|_{L^2(T)}$ satisfy

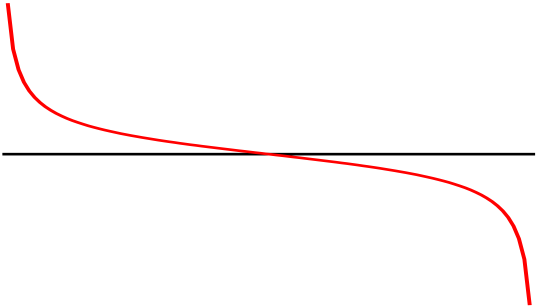
$$\mu_\ell^2 = \sum_{T\in\mathcal{T}_\ell} \mu_\ell(T)^2$$

- $\widetilde{\mu}_\ell \lesssim \eta_\ell = \|\widehat{\Phi}_\ell - \Phi_\ell\| \lesssim \|h_\ell/\varrho_\ell\|_{L^\infty(\Gamma)}^{1/2}\,\widetilde{\mu}_\ell$

- theoretically, $\|h_\ell/\varrho_\ell\|_{L^\infty(\Gamma)}$ measures “how isotropic is $\mathcal{T}_\ell$”
  - bounded for 2D and isotropic mesh-refinement in 3D
  - theoretically unbounded for anisotropic mesh-refinement in 3D
- practically, not visible if $\mathcal{T}_\ell$ fits to singularities of u

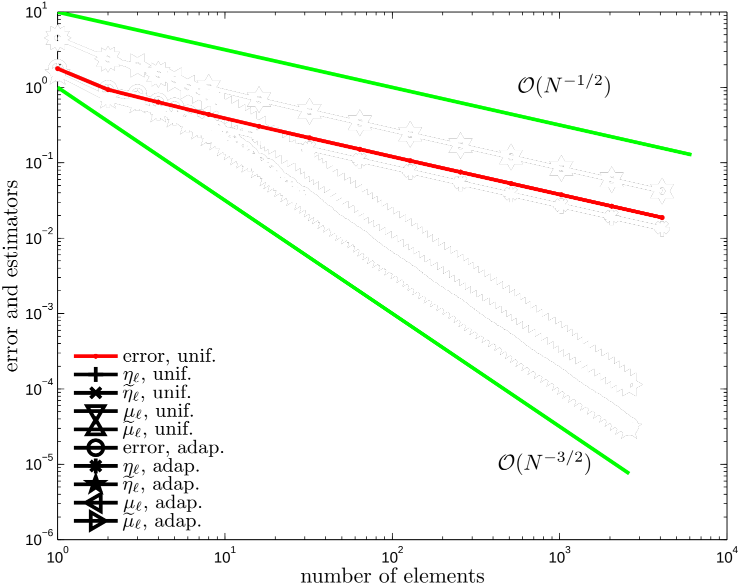
Numerical experiment in 2D

- 1D boundary piece $\Gamma = (-1,1) \times \{0\}$
- $u(x,y) = -2x/\sqrt{1-x^2}$ solve $V\phi(x,y) = -x$



- singularities at $\pm(1,0)$

Numerical experiment in 2D (p=0)



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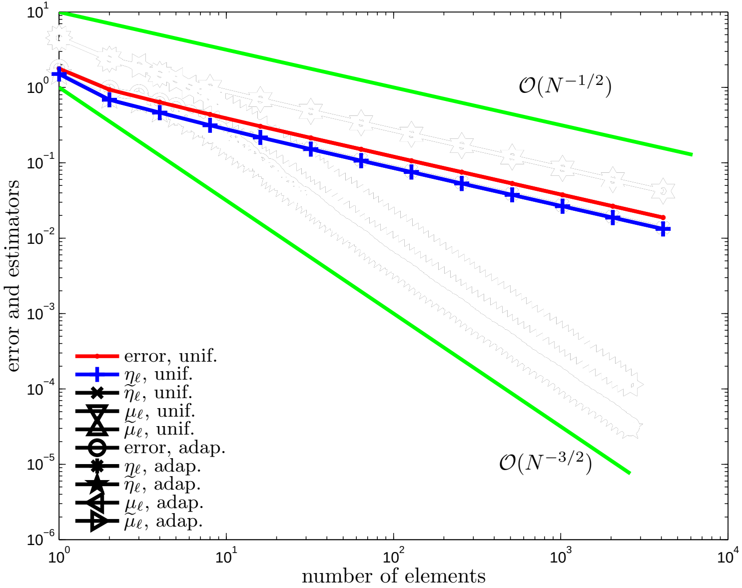

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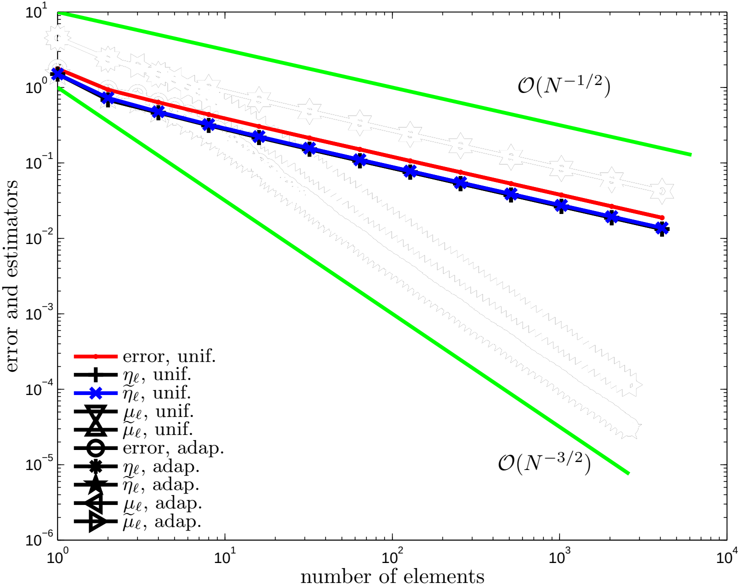

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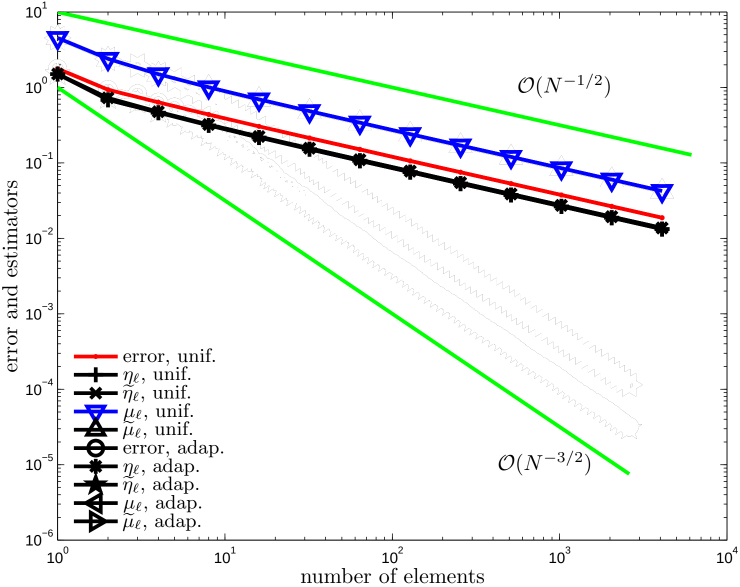
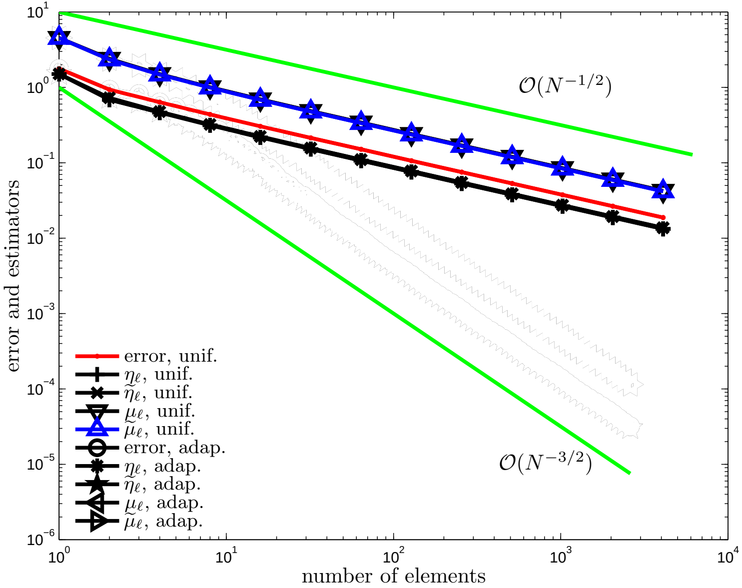
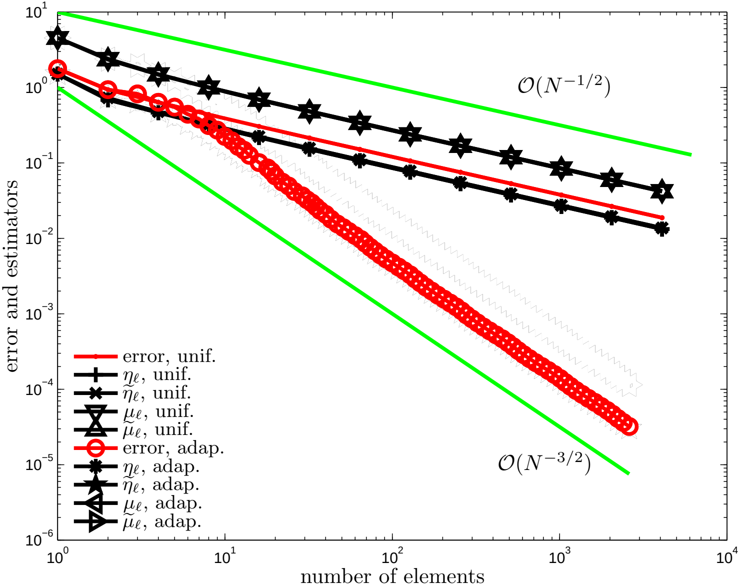


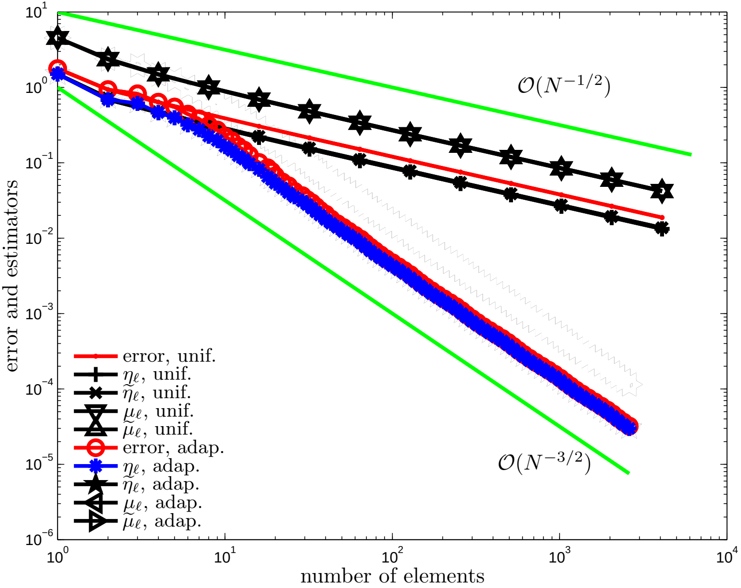
Numerical experiment in 2D (p=0)



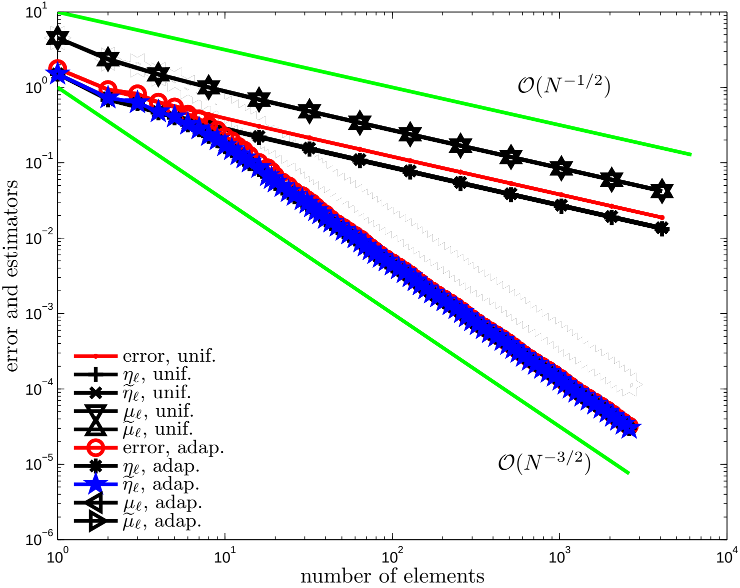
Numerical experiment in 2D (p=0)



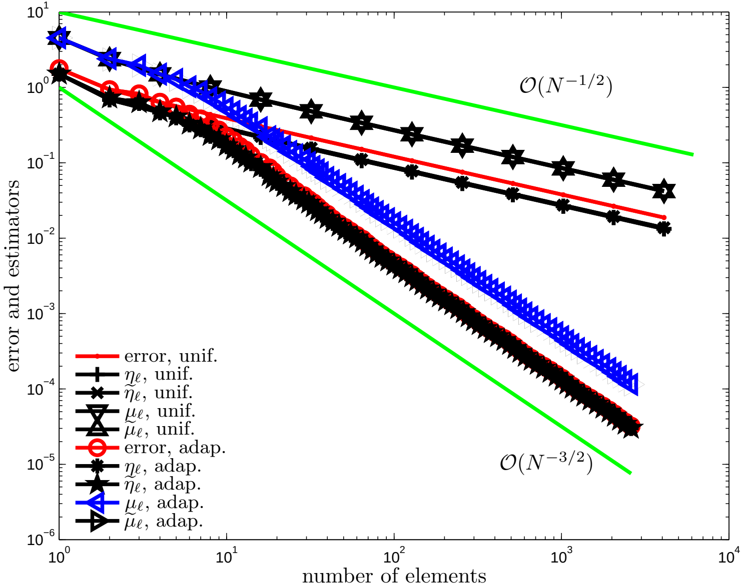
Numerical experiment in 2D (p=0)



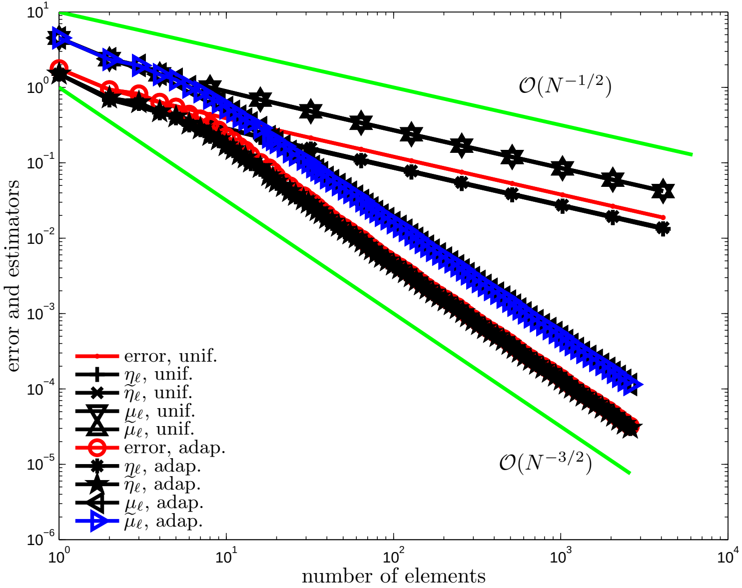
Numerical experiment in 2D (p=0)



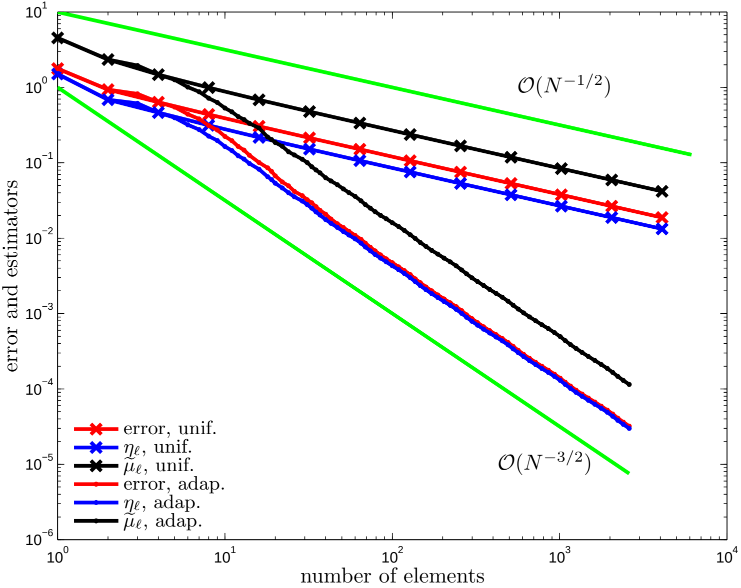
Numerical experiment in 2D (p=0)



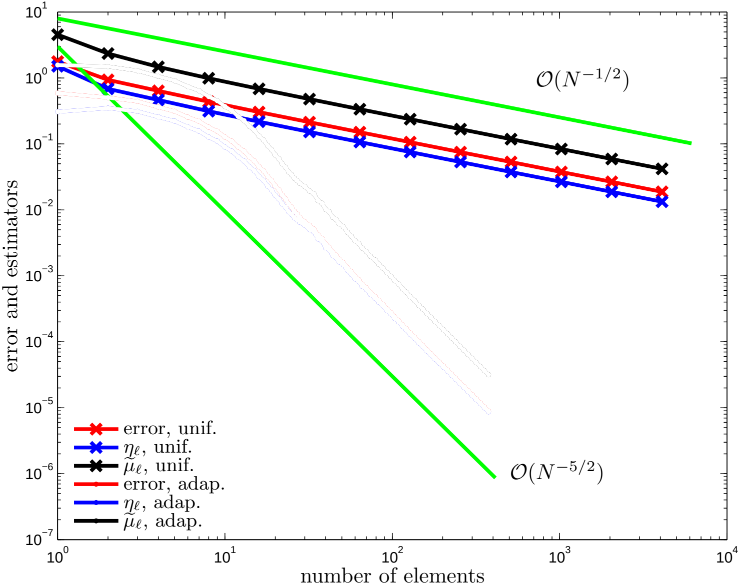
Numerical experiment in 2D (p=0)



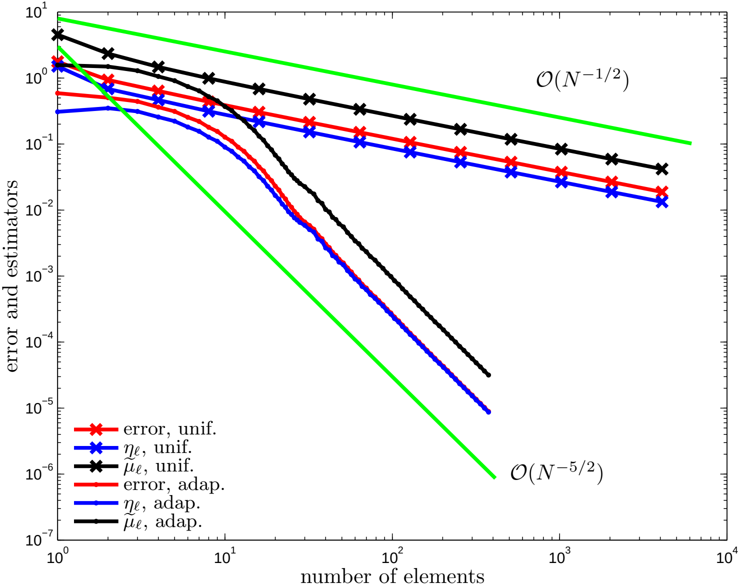
Numerical experiment in 2D (p=0)



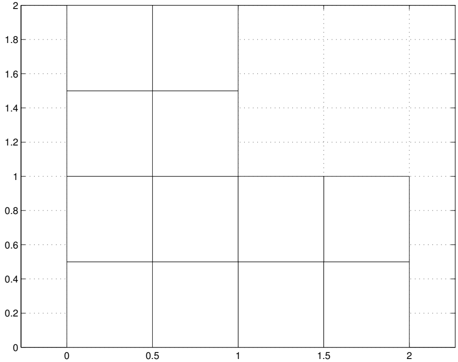
Numerical experiment in 2D (p=1)



Numerical experiment in 2D (p=1)

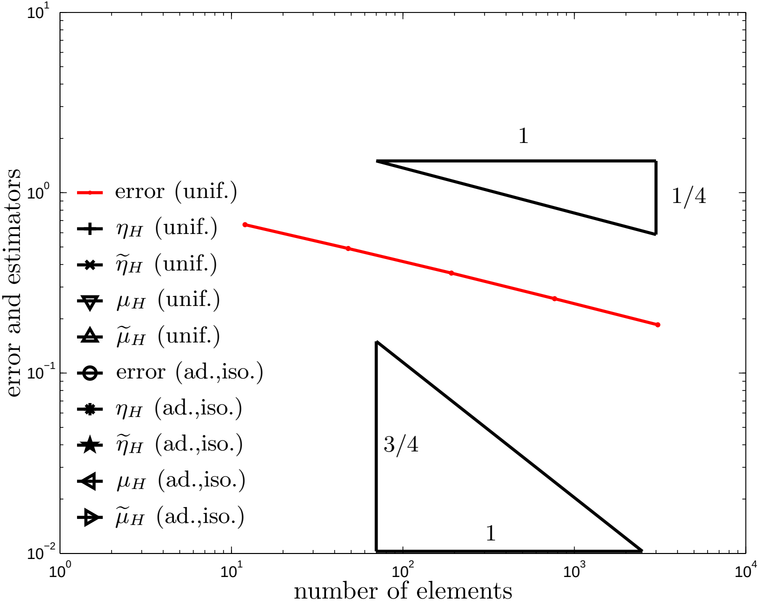


Numerical experiment in 3D

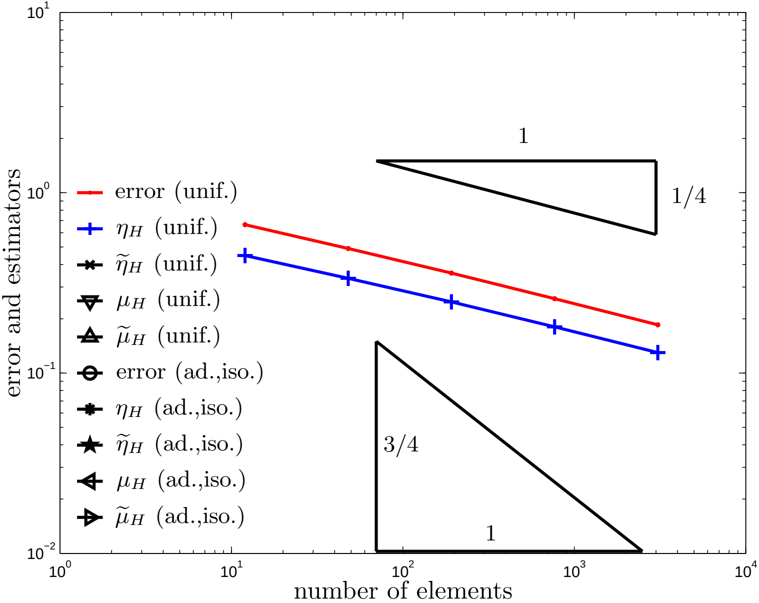


- consider $Vu = 1$ on L-shaped screen $L \times \{0\} \subset \mathbb{R}^3$
- initial mesh $\mathcal{T}_0$ with $N = 12$ elements
- lowest-order BEM $p = 0$

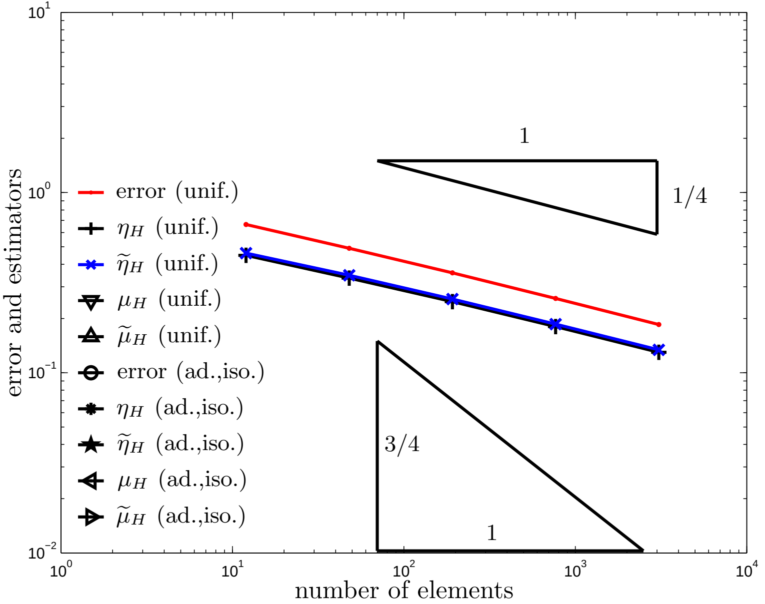
Numerical experiment in 3D (isotropic)



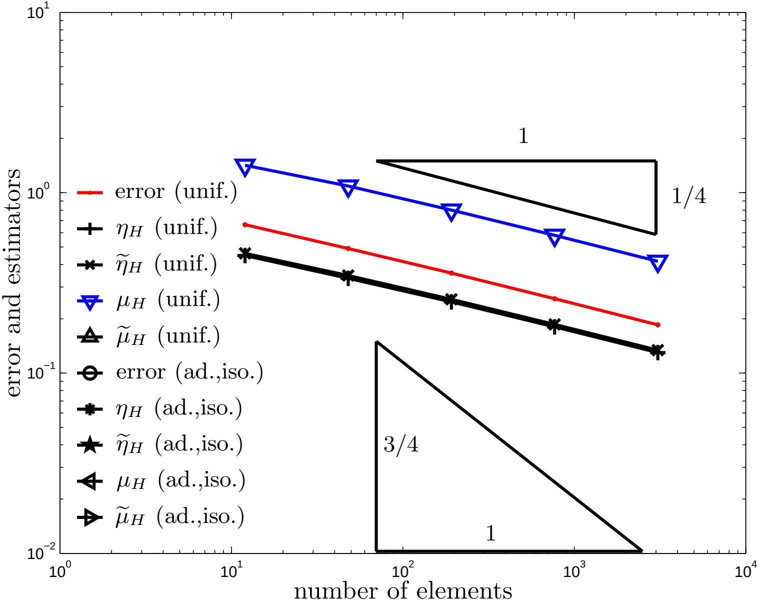
Numerical experiment in 3D (isotropic)



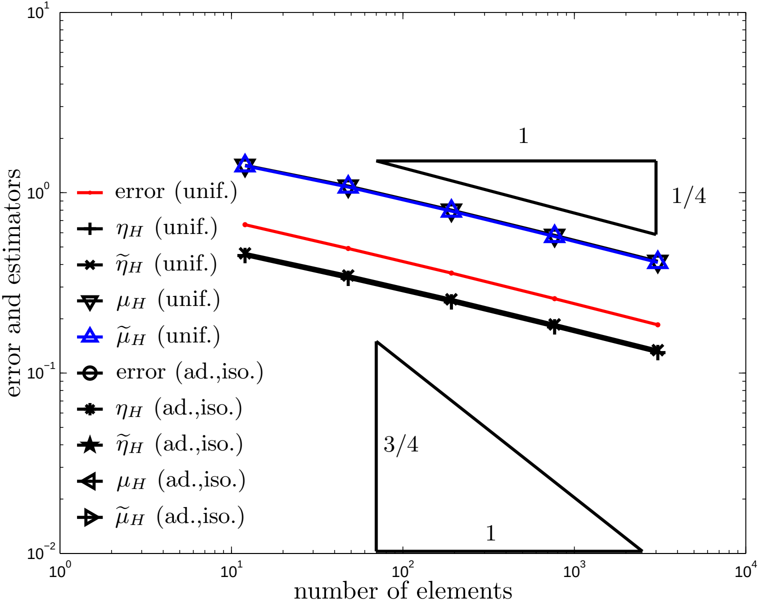
Numerical experiment in 3D (isotropic)



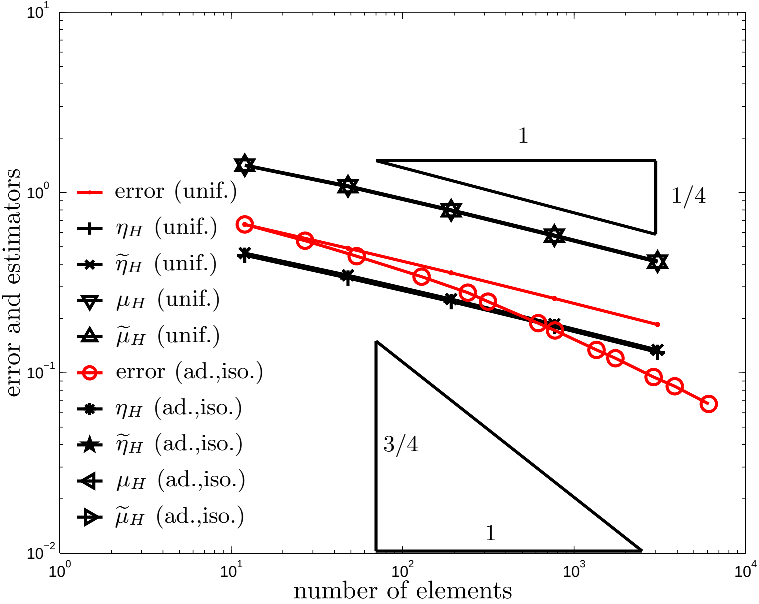
Numerical experiment in 3D (isotropic)



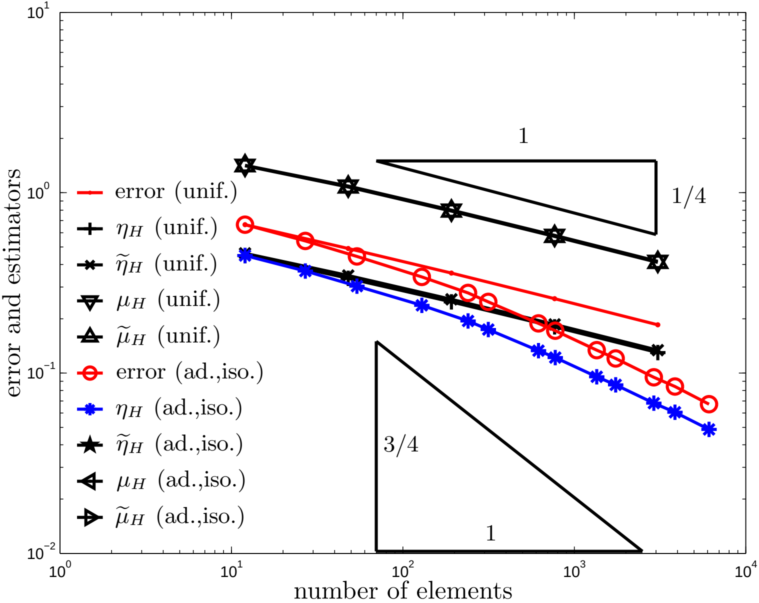
Numerical experiment in 3D (isotropic)



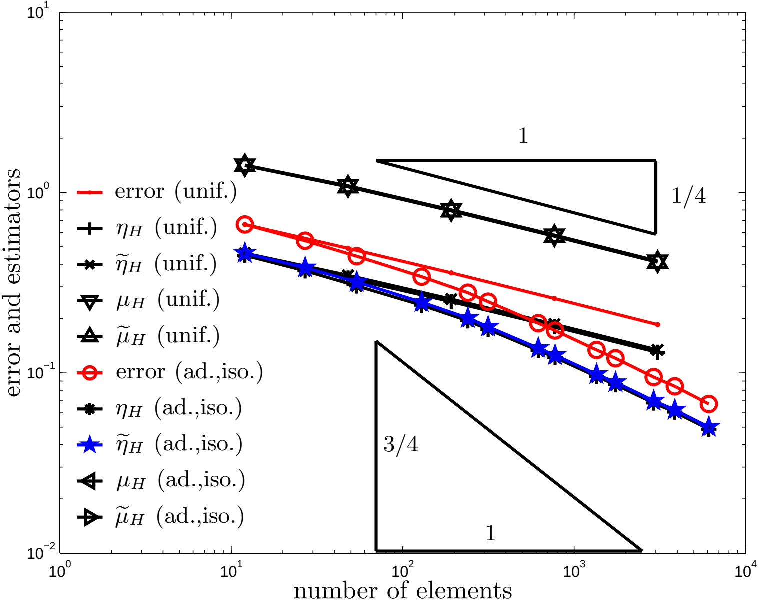
Numerical experiment in 3D (isotropic)



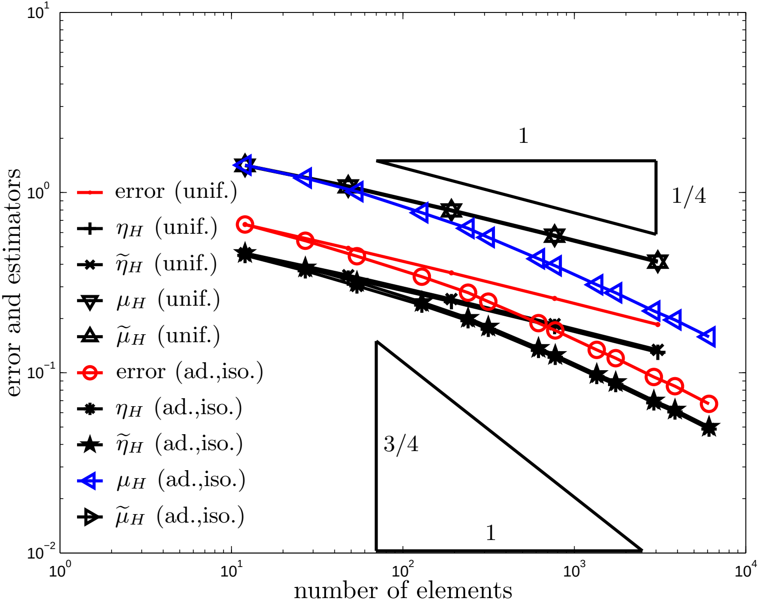
Numerical experiment in 3D (isotropic)



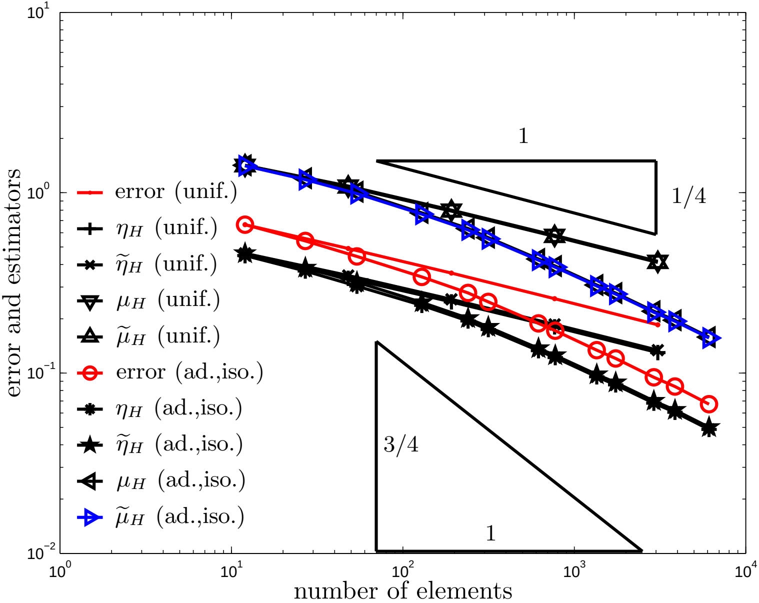
Numerical experiment in 3D (isotropic)



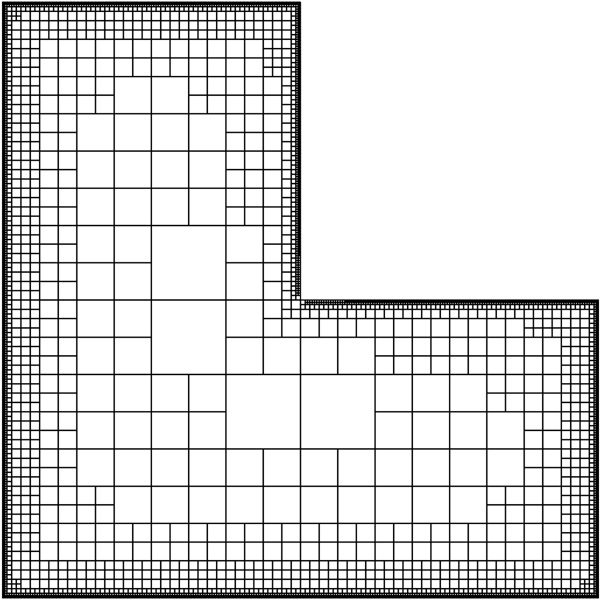
Numerical experiment in 3D (isotropic)



Numerical experiment in 3D (isotropic)



Adaptive Isotropic Mesh

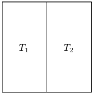


Steering of anisotropic mesh-refinement

- suppose $T \in \mathcal{M}_\ell$
- fix parameter $0 < \tau < 1$
- L^2-orthogonal projection $\Pi_{\texttt{horiz}}$ if T is horizontally refined
- L^2-orthogonal projection $\Pi_{\texttt{vert}}$ if T is vertically refined

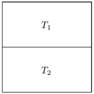
- if: $\|(1 - \Pi_{\texttt{horiz}})\widehat{\Phi}_\ell\|_{L^2(T)} \leq \tau \, \|(1 - \Pi_\ell)\widehat{\Phi}_\ell\|_{L^2(T)}$

$\Longrightarrow$ refine T horizontally



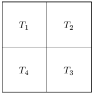
- if: $\|(1 - \Pi_{\texttt{vert}})\widehat{\Phi}_\ell\|_{L^2(T)} \leq \tau \, \|(1 - \Pi_\ell)\widehat{\Phi}_\ell\|_{L^2(T)}$

$\Longrightarrow$ refine T vertically



- else: refine T isotropically

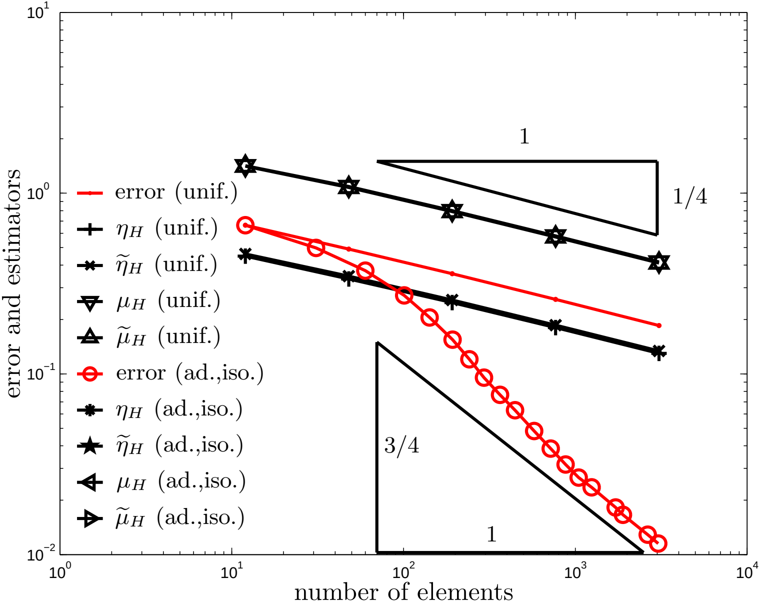
$\Longrightarrow$ $\|(1 - \Pi_{\texttt{iso}})\widehat{\Phi}_\ell\|_{L^2(T)} = 0$



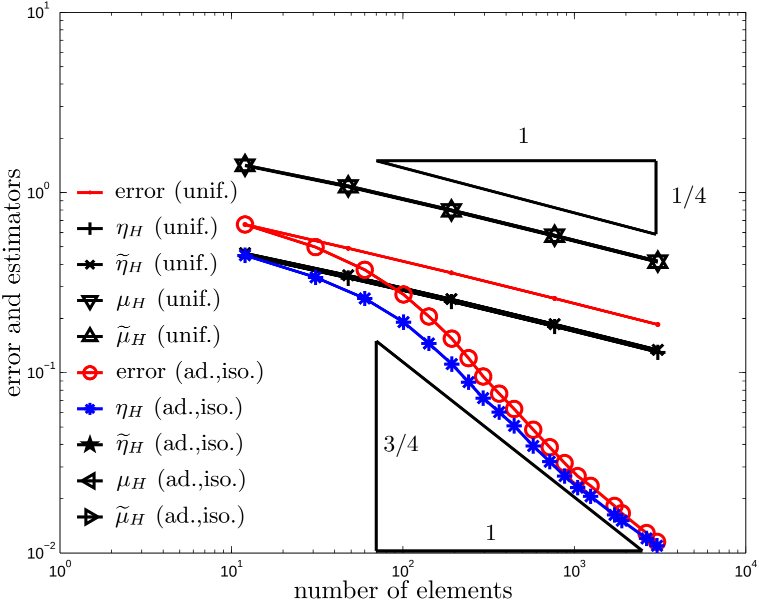
- altogether: $\|\rho_{\ell+1}^{1/2}(1 - \Pi_{\ell+1})\widehat{\Phi}_\ell\|_{L^2(T)} \leq \tau \, \widetilde{\mu}_\ell(T)$

Aurada, Ferraz-Leite, Praetorius: Appl. Numer. Math. 62 (2012)

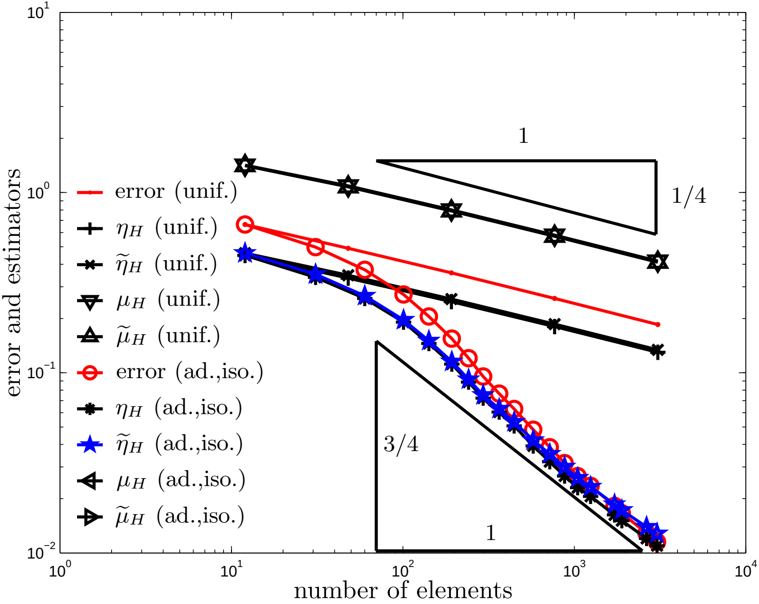
Numerical experiment in 3D (anisotropic)



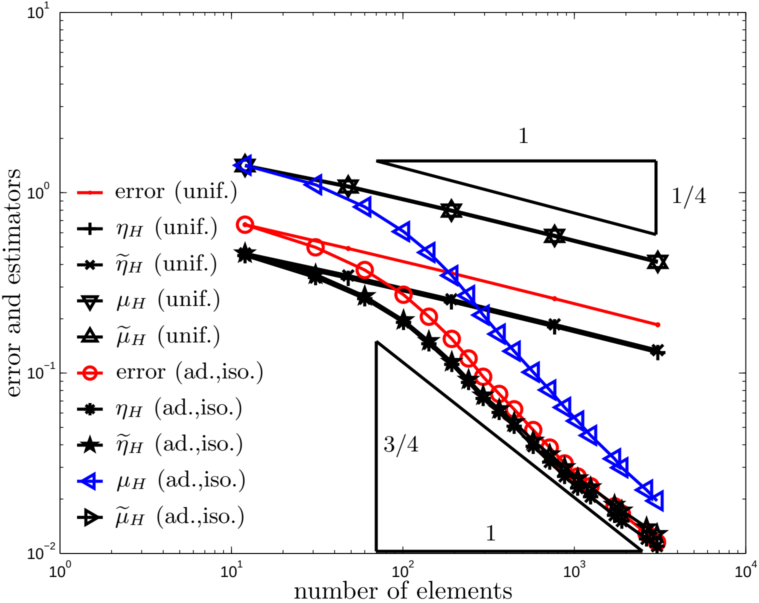
Numerical experiment in 3D (anisotropic)



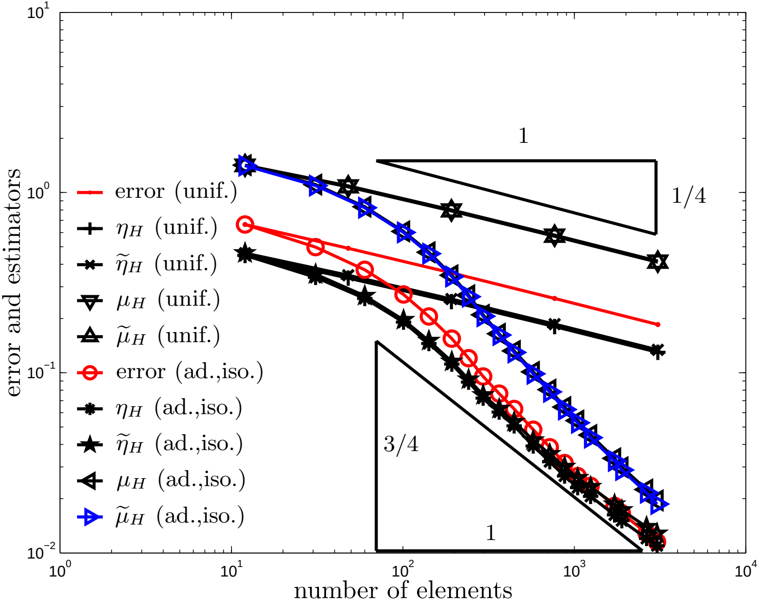
Numerical experiment in 3D (anisotropic)



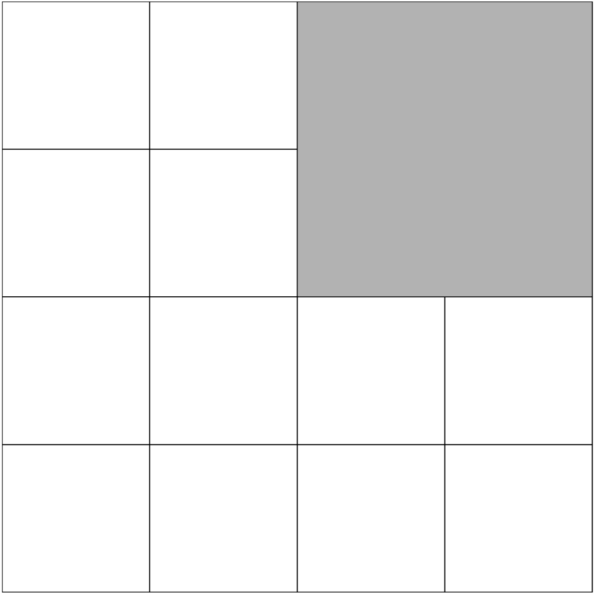
Numerical experiment in 3D (anisotropic)



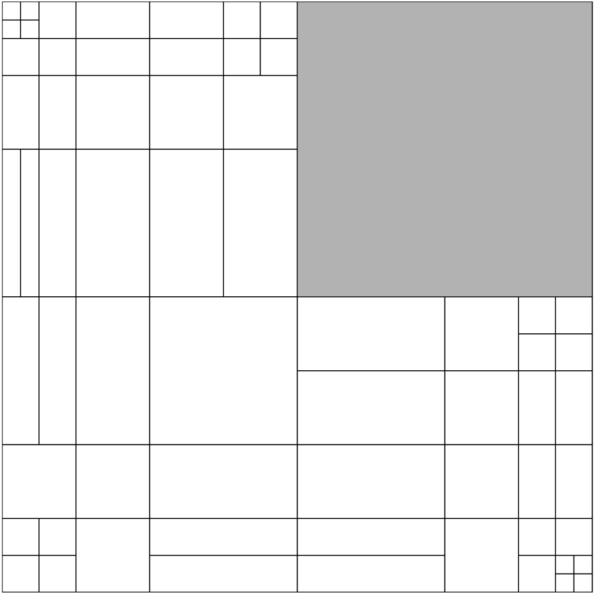
Numerical experiment in 3D (anisotropic)



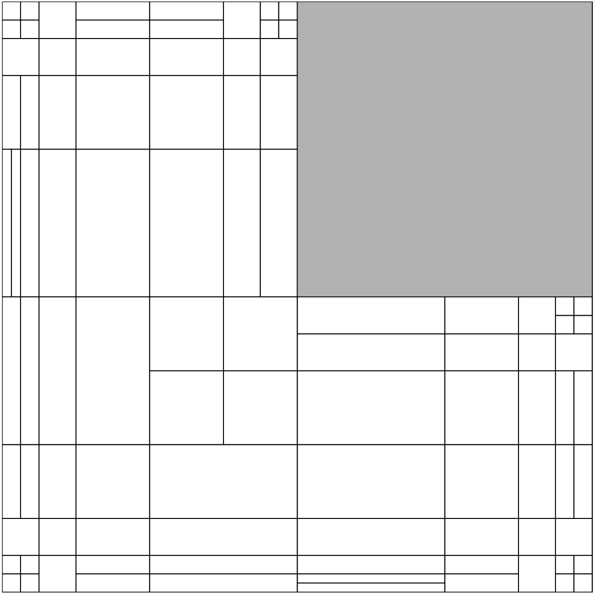
Adaptive anisotropic mesh-refinement



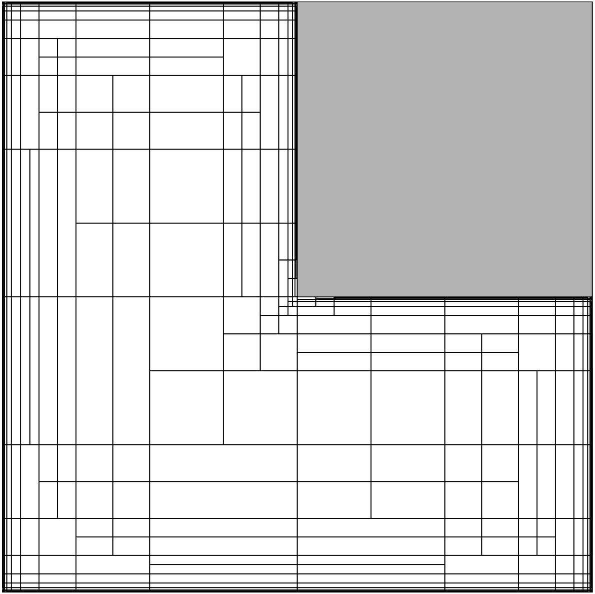
Adaptive anisotropic mesh-refinement



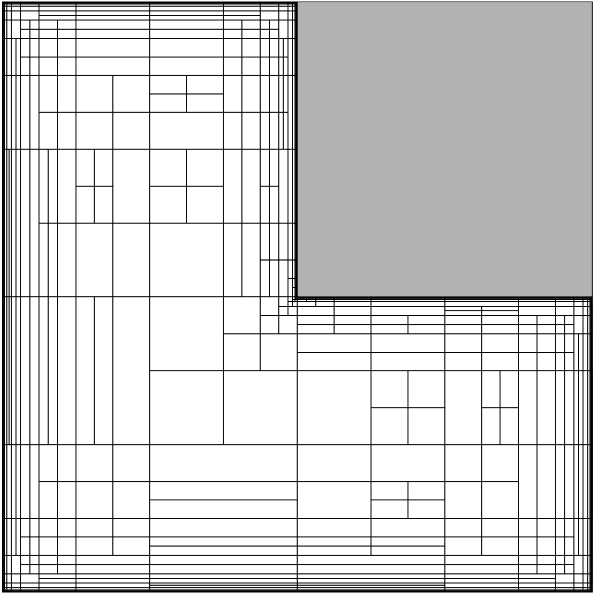
Adaptive anisotropic mesh-refinement



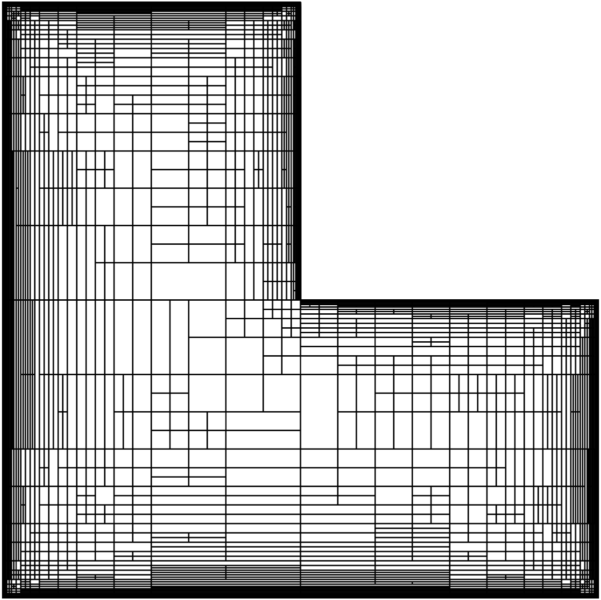
Adaptive anisotropic mesh-refinement



Adaptive anisotropic mesh-refinement



Adaptive anisotropic mesh-refinement



Estimator convergence for h-h/2 ABEM

Theorem (Aurada, Ferraz-Leite, P. '12)

- use $\widetilde{\mu}_\ell$ for marking
- $0 < \theta \le 1$ arbitrary — marking criterion
- $0 \le \tau < 1$ arbitrary — refinement criterion

$\Longrightarrow \quad \eta_\ell \simeq \widetilde{\mu}_\ell \xrightarrow{\ell\to\infty} 0$

- refinement strategy $\Longrightarrow$ $\|\rho_{\ell+1}^{1/2}(1-\Pi_{\ell+1})\widehat{\Phi}_\ell\|_{L^2(T)} \le \tau\,\widetilde{\mu}_\ell(T)$
- Dörfler marking $\Longrightarrow$ $\widetilde{\mu}_{\ell+1} \le \{1-\theta\,(1-\tau)\}\,\widetilde{\mu}_\ell + C\,|\!|\!|\widehat{\Phi}_{\ell+1}-\widehat{\Phi}_\ell|\!|\!|$
- nestedness $\Longrightarrow$ $\widehat{\Phi}_\ell$ converges in $\widetilde{H}^{-1/2}(\Gamma)$

$\Longrightarrow \quad \widetilde{\mu}_{\ell+1} \le q\,\widetilde{\mu}_\ell + o(1) = o(1) \quad$ as $\ell \to \infty$

Aurada, Ferraz-Leite, Praetorius: Appl. Numer. Math. 62 (2012)

Conclusion on h-h/2 ABEM

- $\oplus$ general strategy: almost independent of problem
- $\oplus$ $\eta_\ell \leq |||\phi - \Phi_\ell|||$ with known efficiency constant 1
- $\oplus$ almost no implementational overhead
- $\oplus$ can be computed exactly (up to Galerkin system)
- $\oplus$ good performance of $\widetilde{\mu}_\ell$ on anisotropic meshes

- $\ominus$ $|||\phi - \Phi_\ell||| \lesssim \eta_\ell$ hinges on saturation assumption
- $\ominus$ saturation assumption is hard (impossible?) to guarantee in practice
- $\ominus$ requires Galerkin solution on uniform refinement $\widehat{\mathcal{T}}_\ell$.
- $\ominus$ only analysis for estimator convergence

Residual error estimator

Weighted-residual error estimator 1/2

Variational formulation

- find solution $\phi \in \widetilde{H}^{-1/2}(\Gamma)$ of

$$\langle\!\langle \phi\,,\psi \rangle\!\rangle := \langle V\phi\,,\psi\rangle_{L^2(\Gamma)} = \langle f\,,\psi\rangle_{L^2(\Gamma)} \quad \text{for all } \psi \in \widetilde{H}^{-1/2}(\Gamma)$$

Galerkin formulation

- find solution $\Phi_\ell \in \mathcal{P}^p(\mathcal{T}_\ell)$ of

$$\langle\!\langle \Phi_\ell\,,\Psi_\ell \rangle\!\rangle := \langle V\Phi_\ell\,,\Psi_\ell\rangle_{L^2(\Gamma)} = \langle f\,,\Psi_\ell\rangle_{L^2(\Gamma)} \quad \text{for all } \Psi_\ell \in \mathcal{P}^p(\mathcal{T}_\ell)$$

$$\Longrightarrow \quad \langle f - V\Phi_\ell\,,\Psi_\ell\rangle_{L^2(\Gamma)} = 0 \quad \text{for all} \quad \Psi_\ell \in \mathcal{P}^p(\mathcal{T}_\ell)$$

Weighted-residual error estimator 2/2

Poincaré inequality

- $w \in H^1(\Gamma)$ with $w \perp \mathcal{P}^0(\mathcal{T}_\ell) \quad \Longrightarrow \quad \|w\|_{L^2(\Gamma)} \lesssim \|h_\ell w'\|_{L^2(\Gamma)}$

- $w := f - V\Phi_\ell \in H^1(\Gamma)$ with $w \perp \mathcal{P}^p(\mathcal{T}_\ell) \supseteq \mathcal{P}^0(\mathcal{T}_\ell)$

Reliability (Carstensen, Stephan '95, '96, & Maischak '01)

- $|||\phi - \Phi_\ell||| \simeq \|f - V\Phi_\ell\|_{H^{1/2}(\Gamma)} \lesssim \|h_\ell^{1/2}(f - V\Phi_\ell)'\|_{L^2(\Gamma)} =: \rho_\ell$

- proof requires regularity of $\mathcal{T}_\ell$ for localization
  - 2D: uniformly bounded local mesh-ratio
  - 3D: uniform γ-shape regularity

$\Longrightarrow$ analysis restricted to isotropic meshes

Carstensen, Maischak, Stephan: Numer. Math. 90 (2001)

Reliability & (weak) efficiency

Reliability (Carstensen, Stephan '95, '96, & Maischak '01)

- $|||\phi - \Phi_\ell||| \simeq \|f - V\Phi_\ell\|_{H^{1/2}(\Gamma)} \lesssim \|h_\ell^{1/2}(f - V\Phi_\ell)'\|_{L^2(\Gamma)} =: \rho_\ell$

(weak) Efficiency (Aurada, Feischl, Führer, Karkulik, Melenk, P. '17)

- $\phi \in L^2(\Gamma) \quad \Longrightarrow \quad |||\phi - \Phi_\ell||| \lesssim \rho_\ell \lesssim \|h_\ell^{1/2}(\phi - \Phi_\ell)\|_{L^2(\Gamma)}$

- recall: $|||\phi - \Phi_\ell||| \simeq \|\phi - \Phi_\ell\|_{\widetilde{H}^{-1/2}(\Gamma)} \lesssim \|h_\ell^{1/2}(\phi - \Phi_\ell)\|_{L^2(\Gamma)}$
- converse inverse estimate holds for discrete ϕ

Carstensen, Maischak, Stephan: Numer. Math. 90 (2001)

Aurada, Feischl, Führer, Karkulik, Melenk, Praetorius: Math. Comp. 86 (2017)

Adaptive algorithm (same as before)

- initial mesh $\mathcal{T}_0$
- adaptivity parameter $0 < \theta \leq 1$

For all $\ell = 0, 1, 2, \dots,$ iterate

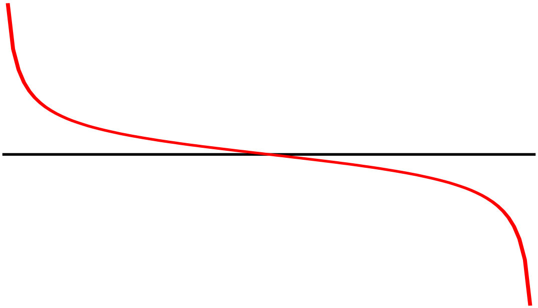
1. compute discrete solution Φ_ℓ for mesh $\mathcal{T}_\ell$
2. compute $\rho_\ell(T) := \text{diam}(T)^{1/2} \, \|(f - V\Phi_\ell)'\|_{L^2(T)}$ for all $T \in \mathcal{T}_\ell$
3. find (essentially minimal) set $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ s.t.

$$\theta \sum_{T \in \mathcal{T}_\ell} \rho_\ell(T)^2 \leq \sum_{T \in \mathcal{M}_\ell} \rho_\ell(T)^2$$

4. refine (at least) marked elements $T \in \mathcal{M}_\ell$ to obtain $\mathcal{T}_{\ell+1}$

Residual error estimator 1/7

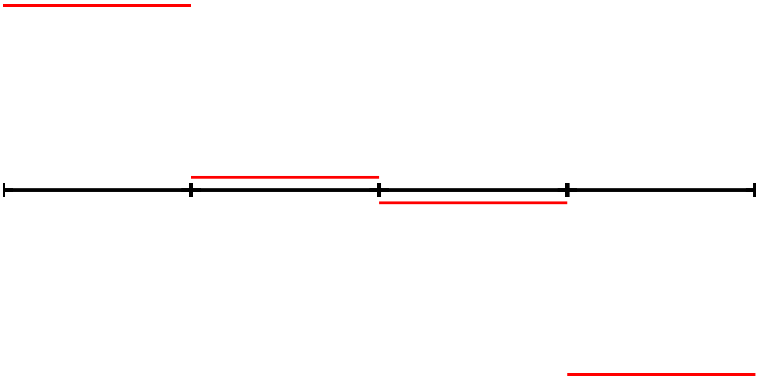
- 1D boundary piece $\Gamma = (-1,1) \times \{0\}$
- $\phi(x,y) = -2x/\sqrt{1-x^2}$ solves $V\phi(x,y) = -x$



- singularities at $\pm(1,0)$

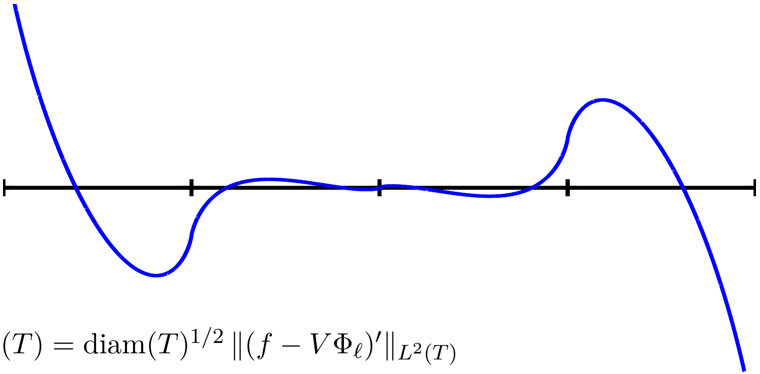
Residual error estimator 2/7

- compute Galerkin solution $\Phi_\ell \in \mathcal{X}_\ell := \mathcal{P}^0(\mathcal{T}_\ell)$



Residual error estimator 3/7

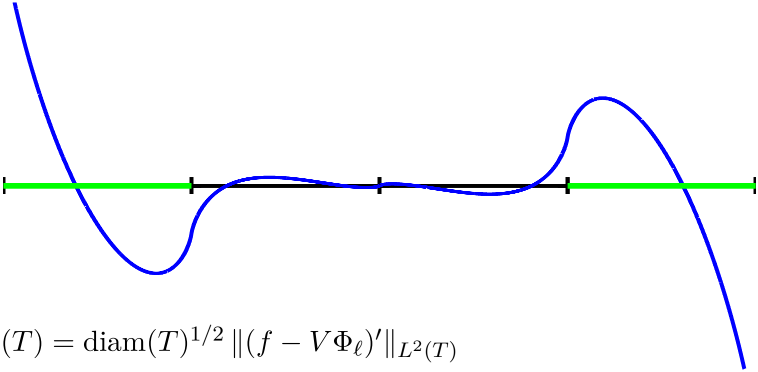
- compute residual on $\mathcal{T}_\ell$



- $\rho_\ell(T) = \mathrm{diam}(T)^{1/2}\, \|(f - V\Phi_\ell)'\|_{L^2(T)}$
- $\displaystyle \frac{1}{4} \sum_{T\in\mathcal{T}_\ell} \rho_\ell(T)^2 \leq \sum_{T\in\mathcal{M}_\ell} \rho_\ell(T)^2$

Residual error estimator 4/7

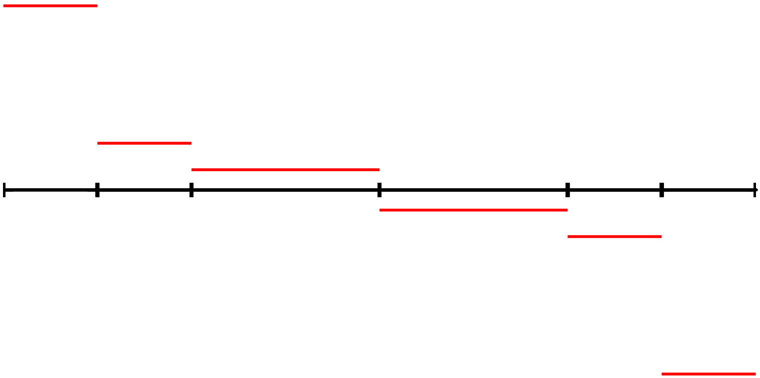
- mark $\mathcal{T}_\ell$ locally, where $\rho_\ell(T)$ is large



- $\rho_\ell(T) = \mathrm{diam}(T)^{1/2}\,\|(f - V\Phi_\ell)'\|_{L^2(T)}$
- $\displaystyle \frac{1}{4}\sum_{T\in\mathcal{T}_\ell}\rho_\ell(T)^2 \le \sum_{T\in\mathcal{M}_\ell}\rho_\ell(T)^2$

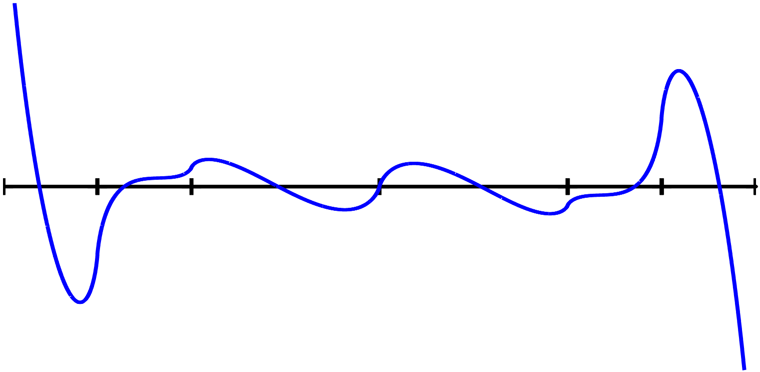
Residual error estimator 5/7

- refine marked elements to get new mesh $\mathcal{T}_\ell$
- compute Galerkin solution $\Phi_\ell \in \mathcal{X}_\ell := \mathcal{P}^0(\mathcal{T}_\ell)$



Residual error estimator 6/7

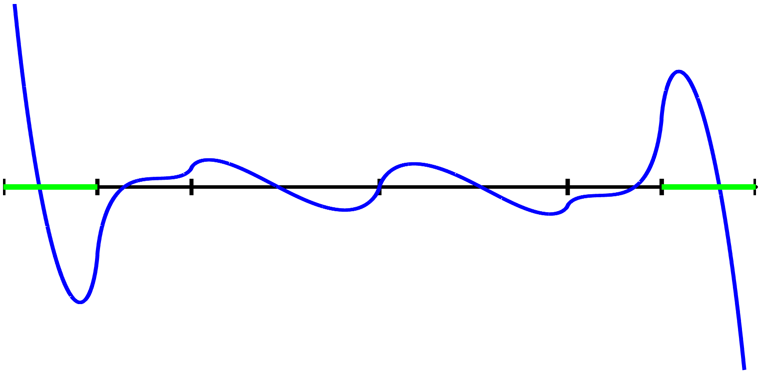
- compute residual on $\mathcal{T}_\ell$



- $\rho_\ell(T) = \mathrm{diam}(T)^{1/2}\, \|(f - V\Phi_\ell)'\|_{L^2(T)}$
- $\dfrac{1}{4} \sum\limits_{T\in\mathcal{T}_\ell} \rho_\ell(T)^2 \le \sum\limits_{T\in\mathcal{M}_\ell} \rho_\ell(T)^2$

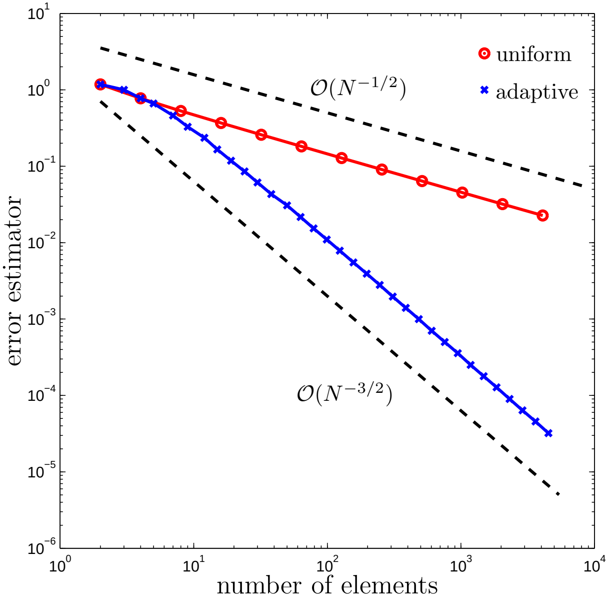
Residual error estimator 7/7

- mark $\mathcal{T}_\ell$ locally, where $\rho_\ell(T)$ is large



- $\rho_\ell(T) = \mathrm{diam}(T)^{1/2}\,\|(f - V\Phi_\ell)'\|_{L^2(T)}$
- $\displaystyle \frac{1}{4}\sum_{T\in\mathcal{T}_\ell}\rho_\ell(T)^2 \le \sum_{T\in\mathcal{M}_\ell}\rho_\ell(T)^2$

ABEM results in...

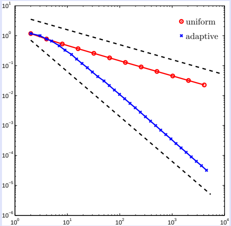


Main theorem on rate optimal BEM adaptivity

Theorem (Feischl, Karkulik, Melenk, P. '13; Gantumur '13)

- ABEM with residual error estimator on isotropic meshes
- $0 < \theta \leq 1$

$\Longrightarrow \quad \exists\, C > 0 \;\exists\, 0 < q < 1 \;\forall\, \ell, n \geq 0 \quad \rho_{\ell+n} \leq C\, q^n\, \rho_\ell$



- $\mathbb{T}_N := \{\mathcal{T} \in \texttt{refine}(\mathcal{T}_0) \,:\, \#\mathcal{T} \leq N\} \cup \{\mathcal{T}_0\}$
- $s > 0$ arbitrary
- $0 < \theta \ll 1$ sufficiently small
- $\mathcal{M}_\ell$ has (essentially) minimal cardinality

$\Longrightarrow \quad \sup_{\ell \in \mathbb{N}_0} (\#\mathcal{T}_\ell)^s \rho_\ell \simeq \sup_{N>0} \big(N^s \min_{\mathcal{T}_{\rm opt} \in \mathbb{T}_N} \rho_{\rm opt} \big) =: \|\rho\|_{\mathbb{A}_s}$

Feischl, Karkulik, Melenk, Praetorius: SINUM 51 (2013)

Gantumur: Numer. Math. 124 (2013)

State-of-the-art proof: Axioms of Adaptivity

$\forall\,\mathcal{T}_H \quad \forall\,\mathcal{T}_h \in \mathtt{refine}(\mathcal{T}_H)$

(A1) $\Big|\Big(\sum_{T\in\mathcal{T}_H\cap\mathcal{T}_h}\rho_h(T)^2\Big)^{1/2} - \Big(\sum_{T\in\mathcal{T}_H\cap\mathcal{T}_h}\rho_H(T)^2\Big)^{1/2}\Big| \le C_{\rm stab}\,\|\Phi_h - \Phi_H\|$

(A2) $\sum_{T\in\mathcal{T}_h\backslash\mathcal{T}_H}\rho_h(T)^2 \le q_{\rm red}\sum_{T\in\mathcal{T}_H\backslash\mathcal{T}_h}\rho_H(T)^2 + C_{\rm red}\,\|\Phi_h - \Phi_H\|^2$

(A3) $\|\Phi_h - \Phi_H\|^2 \le C_{\rm rel}^2\sum_{T\in\mathcal{R}_{Hh}}\rho_H(T)^2$

where $\mathcal{T}_H\backslash\mathcal{T}_h \subseteq \mathcal{R}_{Hh} \subseteq \mathcal{T}_H, \quad \#\mathcal{R}_{Hh} \le C_{\rm rel}\,\#(\mathcal{T}_H\backslash\mathcal{T}_h)$

$\forall\,\ell, N \ge 0 \quad \forall\,\varepsilon > 0$

(A4) $\sum_{k=\ell}^{N}\big(\|\Phi_{k+1} - \Phi_k\|^2 - \varepsilon\rho_k^2\big) \le C_{\rm orth}(\varepsilon)\,\rho_\ell^2$

Carstensen, Feischl, Page, Praetorius: Comput. Math. Appl. 67 (2014)

Axiom (A4): Quasi-orthogonality ✓

$\forall\, \ell, N \geq 0 \quad \forall\, \varepsilon > 0$

(A4) $\displaystyle\sum_{k=\ell}^{N} \big(\|\Phi_{k+1} - \Phi_k\|^2 - \varepsilon \rho_k^2 \big) \leq C_{\rm orth}(\varepsilon)\, \rho_\ell^2$

- Pythagoras thm $\implies \|\phi - \Phi_{k+1}\|^2 + \|\Phi_{k+1} - \Phi_k\|^2 = \|\phi - \Phi_k\|^2$

$$\implies \sum_{k=\ell}^{N} \big(\|\Phi_{k+1} - \Phi_k\|^2 \big) = \sum_{k=\ell}^{N} \big(\|\phi - \Phi_k\|^2 - \|\phi - \Phi_{k+1}\|^2 \big) \leq \|\phi - \Phi_\ell\|^2$$

$$\implies \sum_{k=\ell}^{N} \big(\|\Phi_{k+1} - \Phi_k\|^2 \big) \leq \|\phi - \Phi_\ell\|^2 \lesssim \rho_\ell^2$$ independent of ε, ϕ

- since error estimator is reliable

Feischl, Führer, Praetorius: SINUM 52 (2014)

Local inverse estimate for non-local operators

Theorem (Aurada, Feischl, Führer, Karkulik, Melenk, P '17)

- $\psi \in L^2(\Gamma) \quad \Longrightarrow \quad \|h_\star^{1/2}\nabla V\psi\|_{L^2} \lesssim \|\psi\|_{\widetilde{H}^{-1/2}} + \|h_\star^{1/2}\psi\|_{L^2}$

- in particular, $\Psi_\star \in \mathcal{P}^p(\mathcal{T}_\star) \Longrightarrow \|h_\star^{1/2}\nabla V\Psi_\star\|_{L^2} \lesssim \|\!|\Psi_\star|\!\|$
  - $\|h_\star^{1/2}\Psi_\star\|_{L^2(\Gamma)} \lesssim \|\Psi_\star\|_{\widetilde{H}^{-1/2}(\Gamma)} \simeq \|\!|\Psi_\star|\!\|$

- lowest-order case $\Psi_\star \in \mathcal{P}^0(\mathcal{T}_\ell)$ already in [FKMP '13], [Gantumur '13]

Feischl, Karkulik, Melenk, Praetorius: SINUM 51 (2013)

Gantumur: Numer. Math. 124 (2013)

Aurada, Feischl, Führer, Karkulik, Melenk, Praetorius: Math. Comp. 86 (2017)

Axiom (A1): Stability on non-refined elements

(A1) Stability on non-refined elements, $\mathcal{T}_h \in \texttt{refine}(\mathcal{T}_H)$

$$\Big| \Big(\sum_{T \in \mathcal{T}_H \cap \mathcal{T}_h} \rho_h(T)^2 \Big)^{1/2} - \Big(\sum_{T \in \mathcal{T}_H \cap \mathcal{T}_h} \rho_H(T)^2 \Big)^{1/2} \Big| \lesssim |||\Phi_h - \Phi_H|||$$

- $\rho_h(T)^2 = \mathrm{diam}(T)\, \|\nabla(f - V\Phi_h)\|_{L^2(T)}^2$
- $\sum_{T \in \mathcal{T}_H \cap \mathcal{T}_h} \rho_h(T)^2 = \|h^{1/2}\nabla(f - V\Phi_h)\|_{L^2(\bigcup(\mathcal{T}_H \cap \mathcal{T}_h))}^2$
- inverse triangle inequality + novel inverse estimate

$$\begin{aligned} \mathrm{LHS} &\le \|h^{1/2}\nabla(f - V\Phi_h) - H^{1/2}\nabla(f - V\Phi_h)\|_{L^2(\bigcup(\mathcal{T}_H \cap \mathcal{T}_h))} \\ &\le \|h^{1/2}\nabla V(\Phi_H - \Phi_h)\|_{L^2(\Gamma)} \\ &\lesssim \|\Phi_H - \Phi_h\|_{\widetilde{H}^{-1/2}(\Gamma)} \end{aligned}$$

Cascon, Kreuzer, Nochetto, Siebert: SINUM 46 (2008)

Axiom (A2): Reduction on refined elements

(A2) Reduction on refined elements, $\mathcal{T}_h \in \texttt{refine}(\mathcal{T}_H)$

$$\sum_{T\in\mathcal{T}_h\backslash\mathcal{T}_H} \rho_h(T)^2 \leq q_{\rm red} \sum_{T\in\mathcal{T}_H\backslash\mathcal{T}_h} \rho_H(T)^2 + C_{\rm red}\, |||\Phi_h - \Phi_H|||^2$$

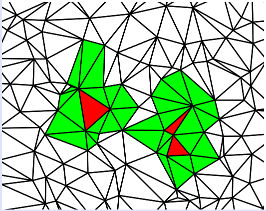
- $\rho_h(T)^2 = \mathrm{diam}(T)\, \|\nabla(f - V\Phi_h)\|^2_{L^2(T)}$
- $\bigcup(\mathcal{T}_h\backslash\mathcal{T}_H) = \bigcup(\mathcal{T}_H\backslash\mathcal{T}_h)$
- $\mathrm{diam}(T') \leq \frac{1}{2}\,\mathrm{diam}(T)$ for $\mathcal{T}_h \ni T' \subsetneqq T \in \mathcal{T}_H$
- triangle inequality + Young ineq. + novel inverse estimate
- $q_{\rm red} \approx 2^{-1/(d-1)}$

Cascon, Kreuzer, Nochetto, Siebert: SINUM 46 (2008)

Axiom (A3): Discrete reliability

(A3) Discrete reliability, $\mathcal{T}_h \in \texttt{refine}(\mathcal{T}_H)$

- exists $\mathcal{R}_{Hh} \subseteq \mathcal{T}_H$ with
  - $\mathcal{T}_H \backslash \mathcal{T}_h \subseteq \mathcal{R}_{Hh}$
  - $\#\mathcal{R}_{Hh} \lesssim \#(\mathcal{T}_H \backslash \mathcal{T}_h)$
  - $\|\Phi_h - \Phi_H\|^2 \le C_{\rm rel}^2 \sum_{T \in \mathcal{R}_{Hh}} \rho_H(T)^2$



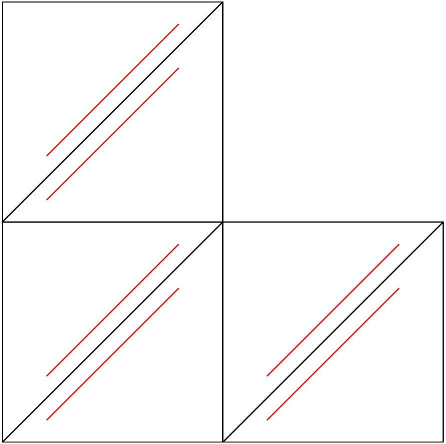
$\Longrightarrow$ "classical" reliability $\|u - \Phi_H\| \le C_{\rm rel}\, \eta_H$

- proof refines usual reliability proof by choice of clever test fct
- $\mathcal{R}_{Hh} = \mathrm{patch}(\mathcal{T}_H \backslash \mathcal{T}_h)$ for BEM

Stevenson: Found. Comput. Math. 7 (2007)

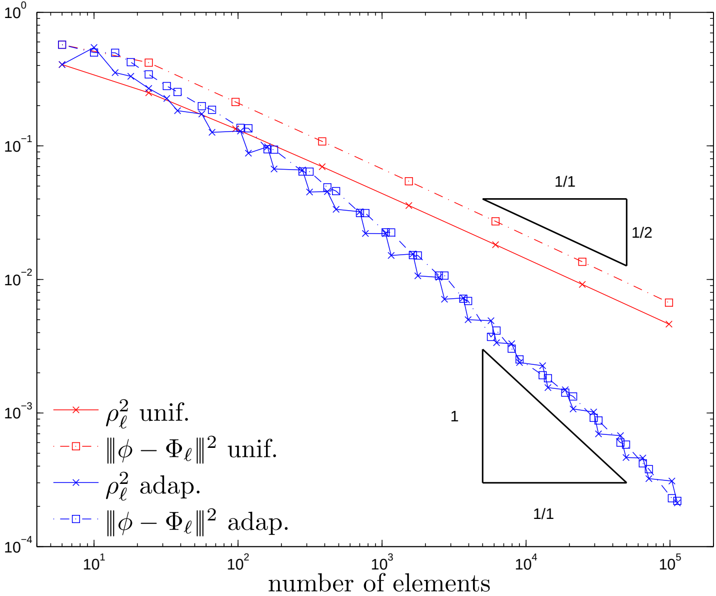
Feischl, Karkulik, Melenk, Praetorius: SINUM 51 (2013)

L-shaped screen

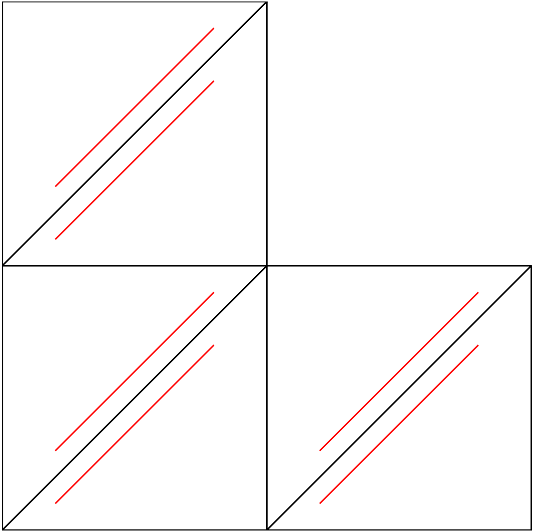


- solve $V\phi = 1$ on Γ with lowest-order BEM $p = 0$

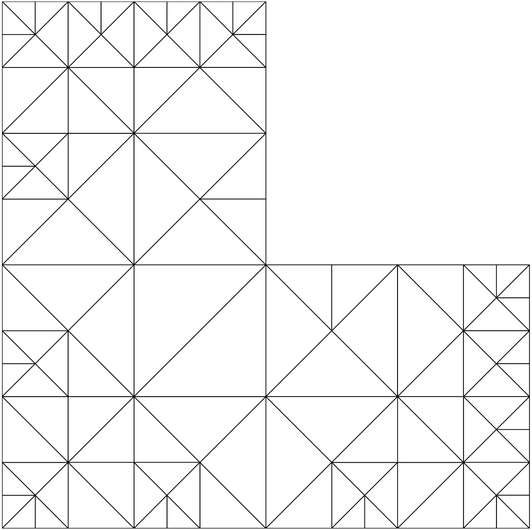
L-shaped screen: errors and estimators



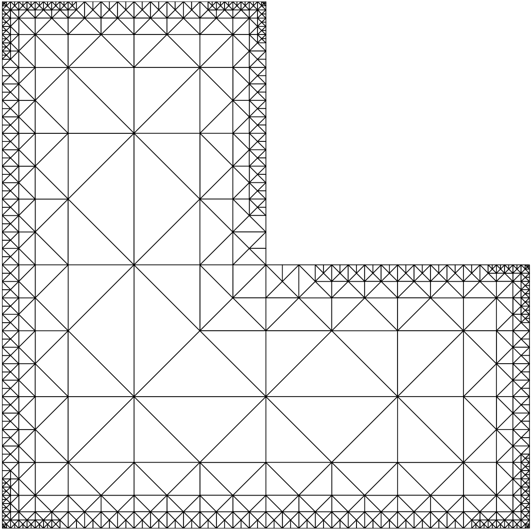
L-shaped screen: adaptive meshes



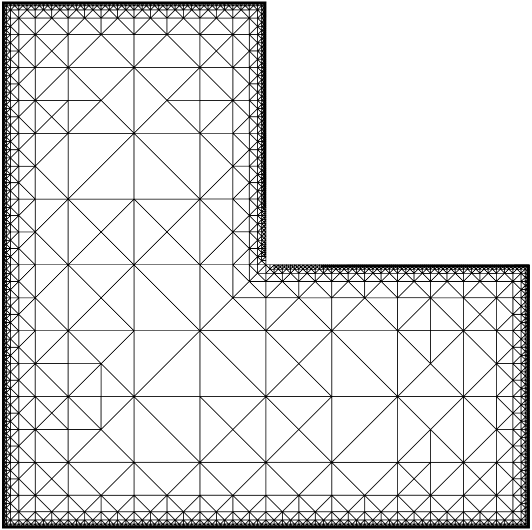
L-shaped screen: adaptive meshes



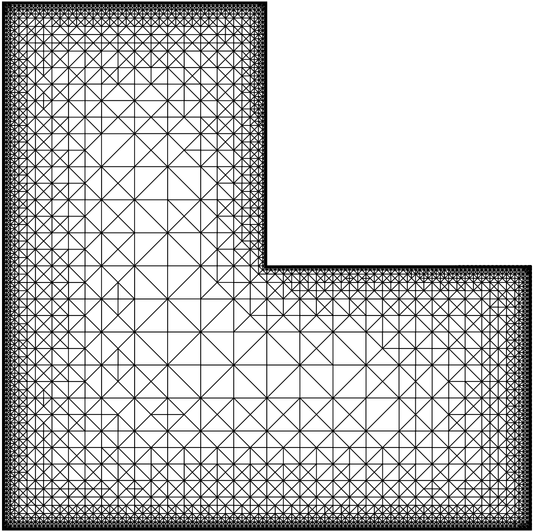
L-shaped screen: adaptive meshes



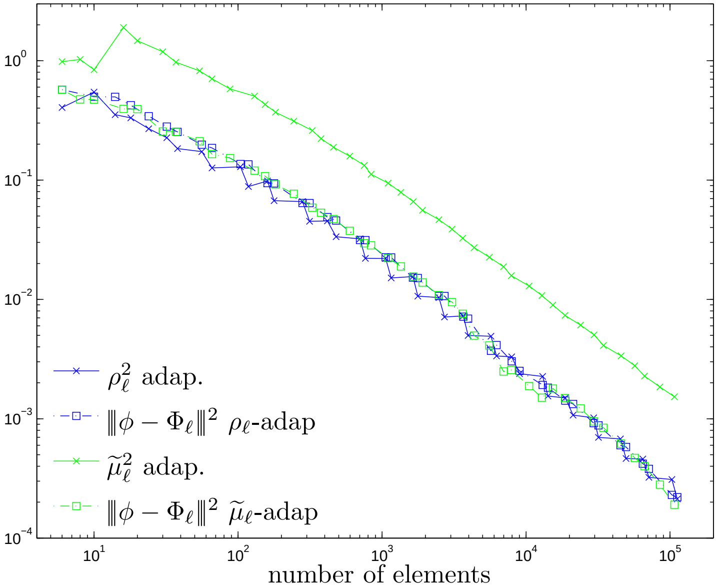
L-shaped screen: adaptive meshes



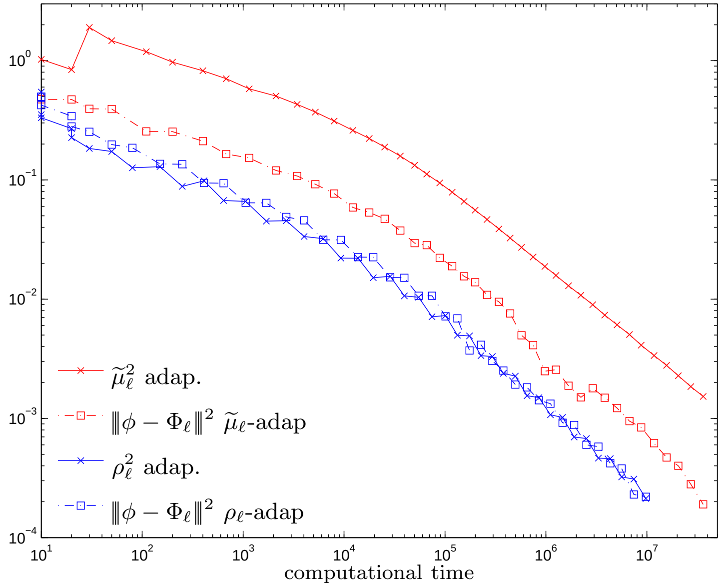
L-shaped screen: adaptive meshes



L-shaped screen: estimator competition



L-shaped screen: estimator competition

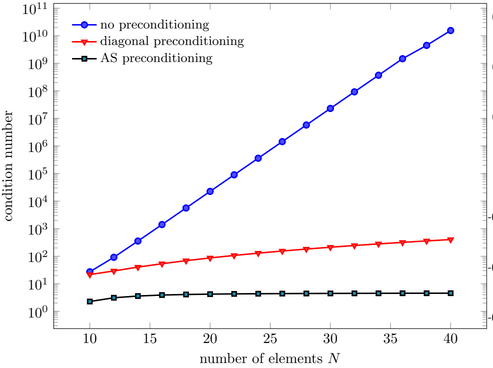


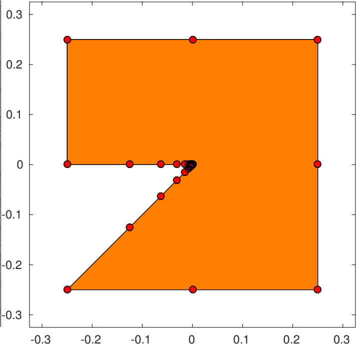
Conclusion on residual-based ABEM

- ⊕ rigorous: $\|\phi - \Phi_\ell\|_{L^2(\Gamma)} \simeq |||\phi - \Phi_\ell||| \lesssim \rho_\ell \lesssim \|h_\ell^{1/2}(\phi - \Phi_\ell)\|_{L^2(\Gamma)}$
- ⊕ rigorous convergence & quasi-optimality analysis
- ⊕ guaranteed convergence for any $0 < \theta \leq 1$
- ⊖ analysis tailored to isotropic meshes
- ⊖ $0 < \theta \ll 1$ from theory is too pessimistic
- ⊖ no analysis for anisotropic mesh-refinement
- ⊖ smart computation of residual estimator is not obvious

Adaptivity and iterative solvers

Preconditioning?!





- no-preconditiong $\Longrightarrow$ $\mathrm{cond}(\boldsymbol{A}) \simeq (h_{\max}/h_{\min})^d \, N^{1/(d-1)}$
- diagonal preconditioning $\Longrightarrow$ $\mathrm{cond}(\boldsymbol{D}^{-1}\boldsymbol{A}) \simeq N^{1/(d-1)}$
- local multilevel additive Schwarz $\Longrightarrow$ $\mathrm{cond}(\boldsymbol{P}^{-1}\boldsymbol{A}) \simeq 1$

Graham, McLean: SINUM 44 (2006)

Führer, Haberl, Praetorius, Schimanko: Numer. Math., accepted (2018)

PCG solver

- applies for $\boldsymbol{A}$ being SPD
- preconditioner $\boldsymbol{P}$ being SPD
- relies only on matrix-vector multiplications with SPD matrix $\boldsymbol{A}$ and $\boldsymbol{P}$
- PCG is an energy method!
  - implicitly standard CG for SPD matrix $\boldsymbol{P}^{-1/2}\boldsymbol{A}\boldsymbol{P}^{-1/2}$
- suppose: $\mathrm{cond}(\boldsymbol{P}^{-1}\boldsymbol{A}) \le C$

$\Longrightarrow$ classical: $\|\Phi_\ell - \Phi_{\ell k}\| \le 2\big(\frac{\sqrt{C}-1}{\sqrt{C}+1}\big)^k \|\Phi_\ell - \Phi_{\ell 0}\|$ a-priori

$\Longrightarrow$ less known: $\|\Phi_\ell - \Phi_{\ell k}\| \le (1 - C^{-1})\, \|\Phi_{\ell k} - \Phi_{\ell(k-1)}\|$ a-posteriori

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Golub, Van Loan: John Hopkins University Press, 2013 (fourth edition)

Extended adaptive algorithm

- initial mesh $\mathcal{T}_0$ with initial guess $\Phi_{00} := 0$
- adaptivity parameters $0 < \theta \le 1$ and $\lambda > 0$

With $(\ell, k) := (0, 0)$, iterate:

1. REPEAT for $k = 1, 2, 3, \dots$
   - do one PCG step to obtain $\Phi_{\ell k}$ from $\Phi_{\ell(k-1)}$
   - compute $\rho_\ell(T, \Phi_{\ell k}) = \mathrm{diam}(T)^{1/2} \, \|(f - V\Phi_{\ell k})'\|_{L^2(T)}$ for all $T \in \mathcal{T}_\ell$

   UNTIL $\|\Phi_{\ell k} - \Phi_{\ell(k-1)}\| \le \lambda \, \rho_\ell(\Phi_{\ell k})$
2. find (essentially minimal) set $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ s.t.
$$\theta \sum_{T \in \mathcal{T}_\ell} \rho_\ell(T, \Phi_{\ell k})^2 \le \sum_{T \in \mathcal{M}_\ell} \rho_\ell(T, \Phi_{\ell k})^2$$
3. refine (at least) all $T \in \mathcal{M}_\ell$ to obtain $\mathcal{T}_{\ell+1}$
4. define $\Phi_{(\ell+1)0} := \Phi_{\ell k}$, update $(\ell, k) \mapsto (\ell + 1, 0)$

Führer, Haberl, Praetorius, Schimanko: Numer. Math., accepted (2018)

Linear convergence

- $\mathcal{Q} := \{(\ell,k) \in \mathbb{N}_0 \times \mathbb{N}_0 \,:\, (\ell,k) \text{ used in algorithm}\}$
- $|(\ell,k)| \quad \rightsquigarrow \quad$ number of overall PCG iterations until $\Phi_{\ell k}$

Theorem (Führer, Haberl, P., Schimanko '18+)

- ABEM with residual error estimator on isotropic meshes
- $0 < \theta \le 1$ arbitrary
- $\lambda > 0$ arbitrary
- quasi-error $\quad \Lambda_{\ell k} := \|\phi - \Phi_{\ell k}\| + \rho_\ell(\Phi_{\ell k})$

$$\Longrightarrow \quad \exists\, C \ge 1 \;\exists\, 0 < q < 1 \;\forall\, (\ell',k') > (\ell,k) \quad \Lambda_{\ell' k'} \le C\, q^{|(\ell',k')| - |(\ell,k)|}\, \Lambda_{\ell k}$$

- algorithm yields linear improvement in each step (PCG or refinement)
- $\Phi_{\ell k}$ solution of last PCG step $\quad \Longrightarrow \quad \|\phi - \Phi_{\ell k}\| \le \Lambda_{\ell k} \simeq \rho_\ell(\Phi_{\ell k})$

Führer, Haberl, Praetorius, Schimanko: Numer. Math., accepted (2018)

Optimal convergence rates

Theorem (Führer, Haberl, P., Schimanko '18+)

- ABEM with residual error estimator on isotropic meshes
- $s > 0$ arbitrary
- $0 < \theta \ll 1$ sufficiently small
- $0 < \tau \ll 1$ sufficiently small
- $\mathcal{M}_\ell$ has (essentially) minimal cardinality

$$\Longrightarrow \quad \sup_{(\ell,k)\in\mathcal{Q}} (\#\mathcal{T}_\ell)^s \Lambda_{\ell k} \simeq \sup_{N>0} \Big(N^s \min_{\mathcal{T}_{\rm opt}\in\mathbb{T}_N} \rho_{\rm opt}(\Phi_{\rm opt})\Big) =: \|\rho\|_{\mathbb{A}_s}$$

$\Longrightarrow$ optimal decay of $\Lambda_{\ell k}$ w.r.t. degrees of freedom

Führer, Haberl, Praetorius, Schimanko: Numer. Math., accepted (2018)

(Almost) optimal computational cost

- costs for one step: $\mathcal{O}\big((\#\mathcal{T}_\ell)\log^2(1+\#\mathcal{T}_\ell)\big)$
- costs for $(\ell',k')\in\mathcal{Q}$: $\mathcal{O}\Big(\sum_{(\ell,k)\le(\ell',k')}(\#\mathcal{T}_\ell)\log^2(1+\#\mathcal{T}_\ell)\Big)$

Theorem

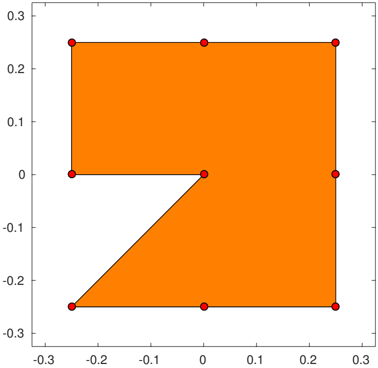
- same assumptions as for optimal rates!
- $s>0$ with $\|\rho\|_{\mathbb{A}_s}<\infty$
- $\varepsilon>0$ arbitrary

$$\Longrightarrow \quad \sup_{(\ell',k')\in\mathcal{Q}}\Big(\sum_{(\ell,k)\le(\ell',k')}(\#\mathcal{T}_\ell)\log^2(1+\#\mathcal{T}_\ell)\Big)^{s-\varepsilon}\Lambda_{\ell'k'}<\infty$$

- not only: convergence with rate s w.r.t. degrees of freedom
- but also: convergence with rate $s-\varepsilon$ w.r.t. costs

Führer, Haberl, Praetorius, Schimanko: Numer. Math., accepted (2018)

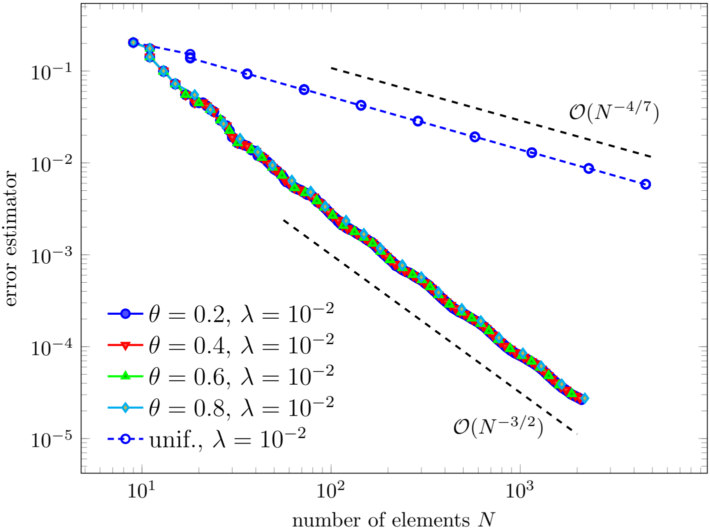
Dirichlet problem on Z-shaped domain



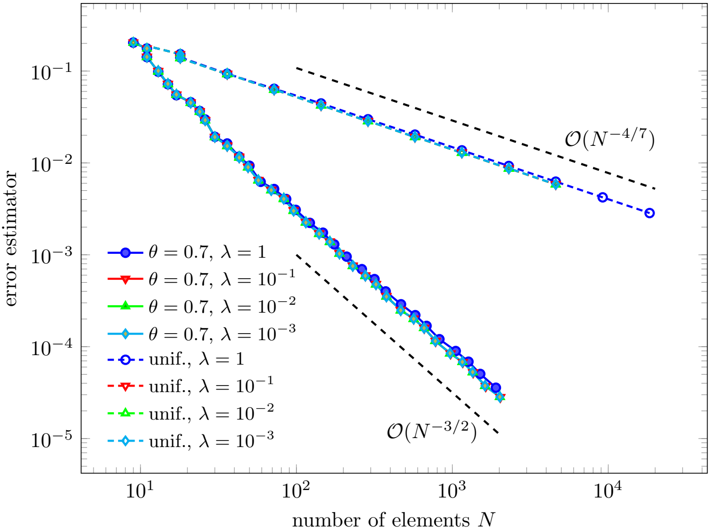
$$\boxed{\begin{aligned} -\Delta u &= 0 \quad \text{in } \Omega \\ u &= g \quad \text{on } \Gamma \end{aligned}} \quad \Longleftrightarrow \quad \boxed{V\phi = (K + 1/2)g \quad \text{on } \Gamma}$$

- prescribed singular solution $u(x) = r^{4/7}\cos(4\xi/7)$ in polar coordinates

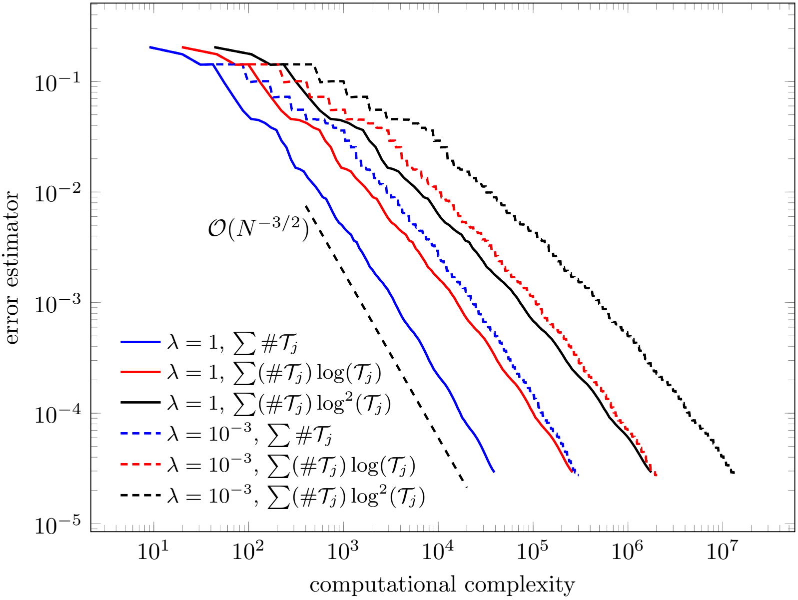
Dependence on θ for $\lambda = 10^{-2}$



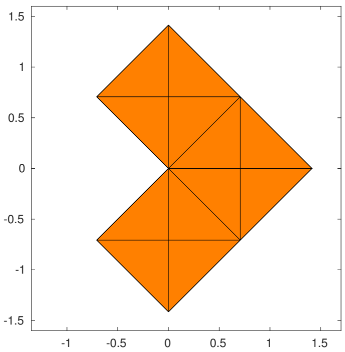
Dependence on λ for $\theta = 0.7$

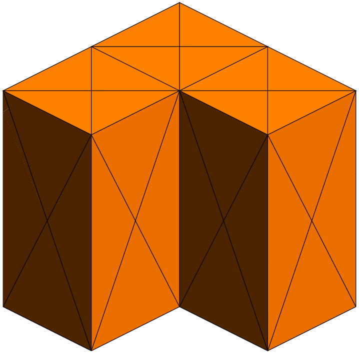


Computational costs



Dirichlet problem on L-shaped domain

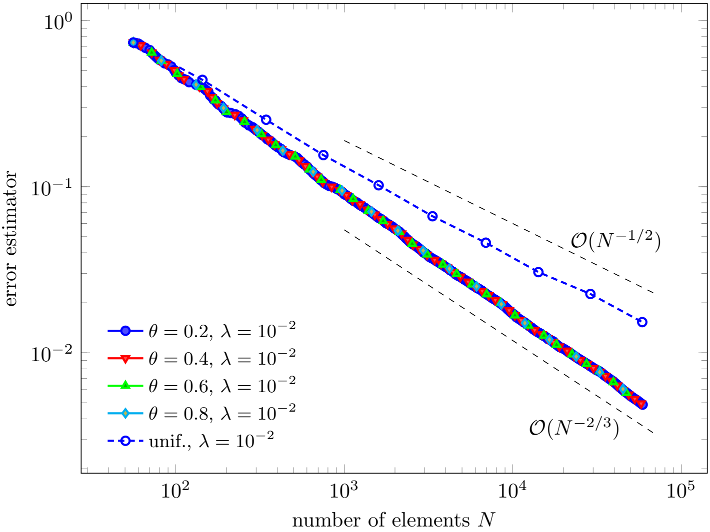




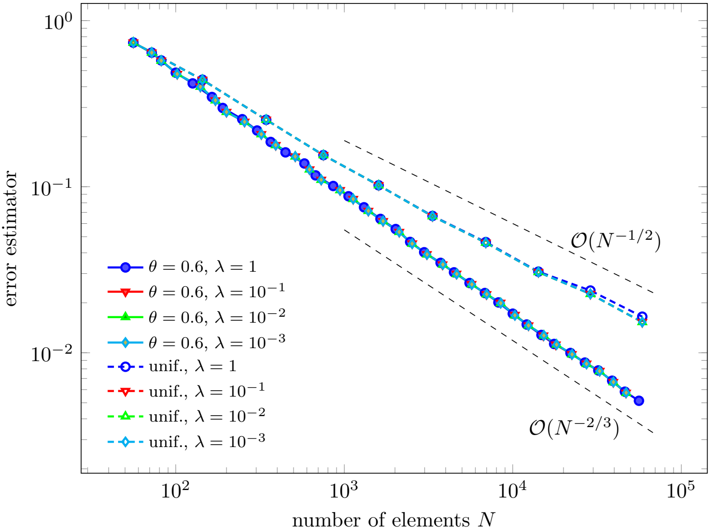
$$\boxed{\begin{aligned} -\Delta u &= 0 \quad \text{in } \Omega \\ u &= g \quad \text{on } \Gamma \end{aligned}} \quad \Longleftrightarrow \quad \boxed{V\phi = (K + 1/2)g \quad \text{on } \Gamma}$$

- prescribed singular solution $u(x) = z\, r^{2/3} \cos(2\xi/3)$

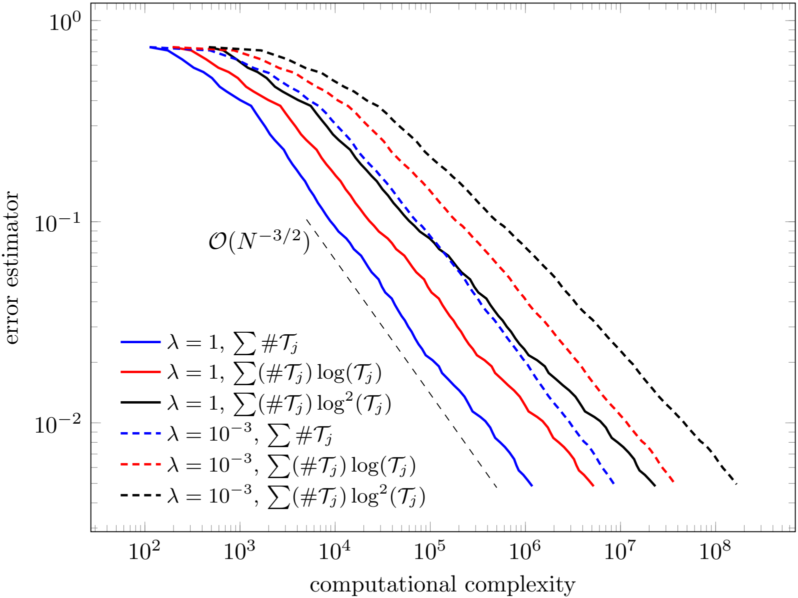
Dependence on θ for $\lambda = 10^{-2}$



Dependence on λ for $\theta = 0.6$



Computational costs



Conclusion on residual-based ABEM + PCG

- $\oplus$ rigorous convergence & quasi-optimality analysis
- $\oplus$ guaranteed convergence for any $0 < \theta \ll 1$ and $\lambda > 0$
- $\oplus$ even almost optimal computational costs
- $\oplus$ costs are optimal in the frame of AFEM!

- $\ominus$ analysis tailored to isotropic meshes
- $\ominus$ $0 < \theta, \lambda \ll 1$ from theory is too pessimistic
- $\ominus$ analysis tailored to PCG solver
- $\ominus$ matrix compression is only treated implicitly

Thanks for listening!

Thomas Führer, Alexander Haberl, Dirk Praetorius, Stefan Schimanko:
Adaptive BEM with inexact PCG solver yields almost optimal computational costs,
Numer. Math., accepted (2018), arXiv:1806.00313

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