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Functional-type a Posteriori Error Estimates for Boundary Element Methods

A 2D toy-project

Daniel Sebastian ■ Särkisaari, 8. August 2018

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Challenges

How a student with an analysis-background tries to do numerics.

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Agenda

- Reminder (BEM)
- Setting
- **Majorant**
- **Minorant**
- First numerical results

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consists of the following parts:

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- ↓ Mathematical model
- ↓ Representation formula
- ↓ Boundary integral equation
- ↓ Boundary elements
- ↓ Discrete equations
- ↓ Solution of the linear system
- ↓ Interpretation

mathematical model

Consider $\Omega \subset \mathbb{R}^N$ a bounded Lipschitz domain

$$\Delta u = 0 \quad \text{in } \Omega$$

$$\gamma_0 u = g \quad \text{on } \Gamma$$

Fundamental solutions for Laplacian are e.g.:

$$G(x,y) = -\frac{1}{2\pi} log|x-y| \quad \text{for } \mathbb{R}^2$$

$$G(x,y) = \frac{1}{4\pi|x-y|} \quad \text{for } \mathbb{R}^3$$

To present the solution of the PDE in the domain by means of boundary potentials, we have for $x \in \mathbb{R}^N \backslash \Gamma$:

$$u(x) = \int_\Gamma \partial_{n(y)} G(x,y) \, [u(y)]_\Gamma \, d\sigma(y) - \int_\Gamma G(x,y) \, [\partial_n u(y)]_\Gamma \, d\sigma(y) \tag{1}$$

where $\quad [v(x)]_\Gamma := v_{|\mathbb{R}^N \backslash \Omega}(x) - v_{|\bar{\Omega}}(x) \quad (x \in \Gamma)$

different ways – make a choice

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1. Supposing $u|\mathbb{R}^N \backslash \Omega = 0$, one obtaines a method called **direct method**:

$$u(x) = -\int_\Gamma \partial_{n(y)} G(x,y) u(y) \, d\sigma(y) + \int_\Gamma G(x,y) \partial_n u(y) \, d\sigma(y) \quad (x \in \Omega). \tag{2}$$

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2. Supposing $[u]_\Gamma = 0$, one obtaines a **single layer representation**

$$u(x) = \int_\Gamma G(x,y) \psi(y) \, d\sigma(y) =: \tilde{V}(\psi)(x) \quad (x \in \Omega) \tag{3}$$

with an unknown density ψ. ($\tilde{V} \colon H^{-1/2}(\Gamma) \to H^1(\Omega)$ bounded linear operator)

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$$u(x) = \int_\Gamma v(y) \partial_{n(y)} G(x,y)\, d\sigma(y) =: \tilde{K}(v)(x) \quad (x \in \Omega) \tag{4}$$

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$\rightarrow$ bounded linear operators

$$V := \gamma_0 \circ \tilde{V} \colon H^{-1/2}(\Gamma) \rightarrow H^{1/2}(\Gamma) \qquad K := \gamma_0 \circ \tilde{K} \colon H^{1/2}(\Gamma) \rightarrow H^{1/2}(\Gamma)$$

1. The idea of a direct method is to approximate the Neumann-data given the Dirichlet-data. The equation to solve here is:

$$V\phi = (\frac{1}{2} + K)g \qquad \text{in } H^{-1/2}(\Gamma)$$

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2. The single layer potential $V\phi(x) := \int_\Gamma G(x,y)\phi(y)\, d\sigma(y)$ is continous across Γ. The equation to solve here is:

$$V\phi = g \qquad \text{in } H^{-1/2}(\Gamma)$$

1. replace boundary Γ by some polygon Γ_h

Discretization

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3. length of segment i denoted by $h_i := |\Gamma_i|$
4. $h := \max_i h_i \qquad h_{min} := \min_i h_i$
5. trial space $S_h^0(\Gamma) :=$span$\{\chi_i\}_{i=1,\ldots,n} \subset H^{-1/2}(\Gamma)$, dim $S_h^0(\Gamma) = n$

Consider $\Omega \subset \mathbb{R}^2$ a polygon

$$\Delta u = 0 \quad \text{in } \Omega$$
$$\gamma_0 u = g \quad \text{on } \Gamma$$

$\Delta : H^1(\Omega) \to H^{-1}(\Omega)$ the Laplace operator,
$\gamma_0 : H^1(\Omega) \to H^{1/2}(\Gamma)$ the Dirichlet trace operator,
g some function determined by the given Dirichlet data $u\big|_\Gamma \in H^{1/2}(\Gamma)$.

Aim at indirect boundary integral equation formulation, based on the single layer potential

$$u(x) = \int_\Gamma G(x, y) w(y) d\Gamma_y, \quad x \in \Omega \subset \mathbb{R}^2$$

where $G(x, y) = -\frac{1}{2\pi} \ln |x - y|$ the fundamental solution in $\mathbb{R}^2$.

Setting – Variational formulation

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Find $\phi \in H^{-1/2}(\Gamma)$ such that

$$\langle \phi', V\phi \rangle = \langle \phi', g \rangle \qquad \forall \phi' \in H^{-1/2}(\Gamma),$$

or respectively, find $\phi_h \in S_h^0(\Gamma)$ *such that*

$$\langle \phi_h', V\phi_h \rangle = \langle \phi_h', g \rangle \qquad \forall \, \phi_h' \in S_h^0(\Gamma). \tag{8}$$

where $\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_{H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)}$ extension of the L^2 scalar product.

Set $u_h = \tilde{V}\phi_h$. Then $\Delta u_h = 0$ but does not fulfill the boundary condition $\gamma_0 u_h = g$ on Γ.

PROBLEMS:

1. Norms of "BEM-typical" non-integer Sobolev-spaces are non-local: "localization techniques" become necessary, e.g. one needs some *interpolation estimate*
2. "Many of residual-based methods use Hölder-, Triangle and Minckowsky-Inequality which lead to overestimation."
3. the error estimates refer to the neumann jump / density function, not towards the global reconstruction

$$F(\phi - \phi_h) = ||\phi - \phi_h||_{H^{-1/2}(\Gamma)} \quad \text{vs.} \quad F(u - v) = ||\nabla(u - v)||_{L^2(\Omega)}$$

It is by triangle inequality

$$\| \nabla(u - u_h) \|_{L^2(\Omega)} \leq \inf_{\substack{v \in H^1(\Omega) \\ \gamma_0 v = g}} \| \nabla(v - u_h) \|_{L^2(\Omega)}.$$

„Shift the problem to the boundary“: *the energy error is equal to the quantity*

$$\varepsilon(u_h) := \inf_{\substack{w \in H^1(\Omega), \\ \gamma_0 w = g - \gamma_0 u_h}} \| \nabla w \|$$

TASK: How to estimate $\varepsilon(u_h)$ via known $e := g - \gamma_0 u_h$ on the boundary with minimal expenditures?

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IDEA:

Find a "good" extension $\Pi e\colon H^{1/2}(\partial\Omega) \to H^1(\Omega)$, the norm of which

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$$\varepsilon(u_h) = \inf_{\substack{w \in H^1(\Omega),\\ \gamma_0 w = g - \gamma_0 u_h}} ||\nabla w||$$

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2. is not expensive to compute (rather than computing the infimum, choose "good"w)

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In the case of BEM one easily computable harmonic extension is:

- Set $e_i := g - \gamma_0 u_h$ on Γ_i and find $\Pi_i \colon H^{1/2}(\Gamma_i) \to H^1(\hat{\Omega}_i)$ s.t.:
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Majorant – idea from Sergey Repin

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$$\varepsilon(u_h) = \inf_{\substack{w \in H^1(\Omega),\\ \gamma_0 w = g - \gamma_0 u_h}} || \nabla w||$$

Then the respective Dirichlet integral is

$$I(e_i) = || \nabla \Pi_i(e_i)||_{\hat{\Omega}_i}$$

and therefore

$$\varepsilon^2(u_h) = \sum_{i=1}^{n} I^2(e_i).$$

Numerical „results“ – majorant with uniform mesh-refinement

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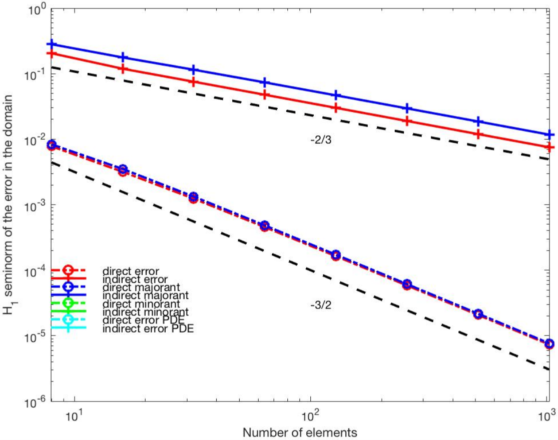

Minorant 1st – only simplicial domains

The simplest minorant (only simplicial domains) results from the Poincaré-type inequality

$$||w||_{L^2(\Gamma)} \leq C(\Gamma, \Omega)||\nabla w||_{L^2(\Omega)}$$

for functions $w \in H^1(\Omega)$ with zero mean trace on Γ.

NOTE: our approximation u_h satisfies $\int_\Gamma (g - \gamma_0 u_h) d\Gamma = 0$ $(1 \in S_h^0(\Gamma)$ and weak formulation)

so we arrive at

$$C^{-1}(\Gamma, \Omega)\, ||e||_\Gamma \leq ||\nabla(u - u_h)||_\Omega$$

where the left hand side is fully computable.

S. Repin, A. Nazarov Exact Constants in Poincaré-Type Inequalities for Functions with Zero Mean Boundary Traces, Math. Meth. Appl. Sci., 2014, vol. 38, no. 15, pp. 3195-3207. and

S. Matculevich, S. Repin. Explicit Constants in Poincaré-Type Inequalities for Simplicial Domains and Application to A Posteriori Estimates. Volume 16, Issue 2 (Apr. 2016

Minorant 2nd

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In Hilbert spaces H we have

$$||\tilde{h}||_H^2 = \max_{h \in H} (2\langle \tilde{h}, h\rangle_H - ||h||_H^2)$$

and therefore defining $e := g - \gamma_0 u_h$

$$\begin{aligned} ||\nabla(u-u_h)||_{L^2(\Omega)} &= \max_{\tau_0 \in L^2(\Omega)} 2\int_\Omega \nabla(u-u_h)\cdot\tau_0 - ||\tau_0||^2_{L^2(\Omega)} \\ &\geq \max_{\tau_0 \in D_0(\Omega)} 2\int_\Omega \nabla(u-u_h)\cdot\tau_0 - ||\tau_0||^2_{L^2(\Omega)} \\ &\geq 2\int_\Gamma e\,\tau_0\cdot n - ||\tau_0||^2_{L^2(\Omega)} \end{aligned}$$

It holds equality if $\tau_0 = \nabla(u-u_h) \in D_0 = H(div;\Omega;div=0)$.

$$||\nabla(u-u_h)||_{L^2(\Omega)} \geq 2\int_\Gamma e\ \mathrm{curl}_z W\cdot n - ||\nabla W||^2_{L^2(\Omega)}$$

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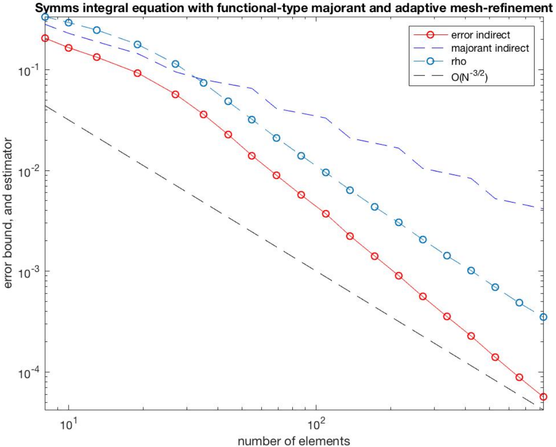
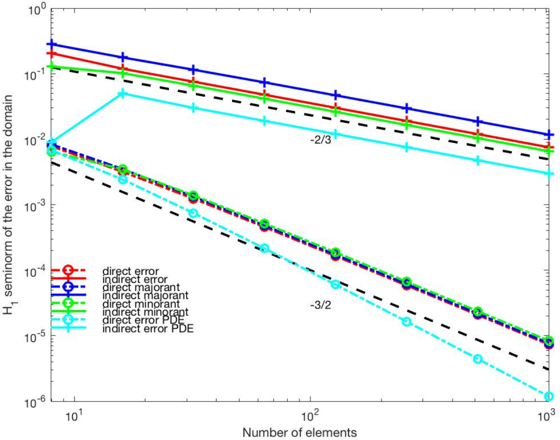
Back to the Majorant!

Numerical „results“ – majorant with adaptive mesh-refinement



Numerical „results“ – all with uniform mesh-refinement



Ways out? – thanks for discussions this week

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- use Green's second identity (for direct method) to compute H^1 seminorm of $u - u_h$

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- use Green's second identity (for direct method) to compute H^1 seminorm of $u - u_h$
- use P^2 elements for interpolating the „error nodes":

$$||\nabla(u - I_h u)||_{L^2(T)} \leq h_T||D^2 u||_{L^2(T)}$$

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THANKS FOR YOUR ATTENTION!

Literature

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