# Optimal Control of Dissipative Solids via Vanishing Viscosity

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What are rate-independent systems and

what are vanishing viscosity solutions?

# Rate-independent systems

- Rate-independent evolutions are driven by external forces on a time scale much slower than their internal scale.
- When the external loadings undergo a time rescaling, the solutions to the rescaled system are the rescaled solutions of the original system.
- examples: dry friction, plasticity, fracture

A. Mielke, T. Roubíček. **Rate-Independent Systems.** Springer, 2015.

# Rate-independent systems

*For an initial value* $z_0 \in \mathcal{Z}$ *find a curve* $z : [0, T] \to \mathcal{Z}$ *such that*

$$0 \in \partial\mathcal{R}(\dot{z}(t)) + D_z\mathcal{I}(\ell(t), z(t)) \text{ f.a.a. } t \in [0, T], \quad z(0) = z_0.$$

- $z \in \mathcal{Z}$ — *state variable* in the *state space*
- $\mathcal{I} : \mathcal{Z}^* \times \mathcal{Z} \to \mathbb{R}$, $\mathcal{I}(\ell, z) = \mathcal{I}_0(z) - \langle \ell, z \rangle$ — *potential energy*
- $\ell \in \mathrm{W}^{1,\infty}(0, T; \mathcal{Z}^*)$ — *external load*
- $\mathcal{R} : \mathcal{Z} \to [0, \infty)$ convex, l.s.c., pos. 1-homogeneous (e.g.: a norm on $\mathcal{Z}$) — *dissipation potential*
- $\partial\mathcal{R} : \mathcal{Z} \rightrightarrows \mathcal{Z}^*$ — *convex subdifferential*
- $D_z\mathcal{I}(\ell, \cdot) : \mathcal{Z} \to \mathcal{Z}^*$ — *Fréchet-differential* of $z \mapsto \mathcal{I}(\ell, z)$

# Rate-independent systems

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interpretations:

- force balance in $\mathcal{Z}^*$:
  dissipative force $\partial\mathcal{R}(\dot{z})$ versus potential restoring force $-D_z\mathcal{I}(\ell, z)$
- if $\mathcal{R}(w) = \frac{1}{2}\|w\|_{\mathcal{Z}}^2$, we would have to solve the gradient flow

$$\dot{z}(t) = -D_z\mathcal{I}(\ell(t), z(t))$$

# Vanishing viscosity solutions

problem: for non-convex $\mathcal{I}$, solutions may have jumps

strategy:

- approximate the system by adding artificial viscosity:

assume $\mathcal{Z} \Subset \mathcal{V} \subset \mathcal{X}$ and $\mathcal{R}(z) = C\|z\|_{\mathcal{X}}$; for $\varepsilon > 0$, set

$$\mathcal{R}_\varepsilon(z) := \mathcal{R}(z) + \tfrac{\varepsilon}{2}\|z\|_{\mathcal{V}}^2$$

and solve

$$0 \in \partial\mathcal{R}_\varepsilon(\dot{z}_\varepsilon(t)) + D_z\mathcal{I}(\ell(t), z_\varepsilon(t)) \text{ a.e.}, \quad z_\varepsilon(0) = z_0$$

1. obtain a sequence of continuous solutions $(z_\varepsilon)_{\varepsilon>0}$
2. show existence of a converging subsequence for $\varepsilon \to 0$
3. find out in which way the limit solves the original problem

# Existence of viscous solutions - assumptions

$\mathcal{I}(\ell, z) := \frac{1}{2}\langle Az, z\rangle_{\mathcal{Z}} + \mathcal{F}(z) - \langle \ell, z\rangle_{\mathcal{V}}$ for $z \in \mathcal{Z}$, where

- $A : \mathcal{Z} \to \mathcal{Z}^*$ is linear, bounded, self-adjoint and coercive
- $\mathcal{F} \in \mathrm{C}^2(\mathcal{Z}, \mathbb{R}_{>0})$ fulfills
  - $D_z\mathcal{F} : \mathcal{Z} \to \mathcal{V}^*$ is weakly continuous and $D_z\mathcal{F} \in \mathrm{C}^1(\mathcal{Z}, \mathcal{V}^*)$,
  - $\exists C > 0, q \geq 1\, \forall z, v \in \mathcal{Z} : \; \|D_z^2\mathcal{F}(z)v\|_{\mathcal{V}^*} \leq C(1 + \|z\|_{\mathcal{Z}}^q)\|v\|_{\mathcal{Z}}$

## Proposition (Existence of viscous solutions)

*There is a solution* $z_\varepsilon \in W^{1,1}([0,T],\mathcal{V})$ *of the viscous system, fulfilling the energy dissipation balance:*

$$\int_s^t \mathcal{R}_\varepsilon(\dot{z}_\varepsilon(r)) + \mathcal{R}_\varepsilon^*(-D_z\mathcal{I}(\ell(r),z_\varepsilon(r)))\,\mathrm{d}r + \mathcal{I}(\ell(t),z_\varepsilon(t))$$
$$= \mathcal{I}(\ell(s),z_\varepsilon(s)) + \int_s^t \partial_r\mathcal{I}(\ell(r),z_\varepsilon(r))\,\mathrm{d}r \qquad \textit{for all } 0 \le s \le t \le T.$$

## Proof.

Mielke, Rossi, Savaré. **Nonsmooth analysis of doubly nonlinear evolution equations.** 2013.

▸ show details

There is a constant $C > 0$ such that for all $\varepsilon > 0$ and all solutions $z_\varepsilon$, it holds:

$$\|z_\varepsilon\|_{L^\infty(0,T;\mathcal{Z})} + \sqrt{\varepsilon}\|z_\varepsilon\|_{H^1(0,T;\mathcal{V})} \le C.$$

There is a sequence of solutions fulfilling

$$\sup_{\varepsilon>0}\left(\|z_\varepsilon\|_{W^{1,1}(0,T;\mathcal{Z})} + \varepsilon\|z_\varepsilon\|_{W^{1,\infty}(0,T;\mathcal{V})}\right) \le C.$$

1. Rate-independent systems and vanishing viscosity solutions
2. Existence of viscous solutions
3. Vanishing viscosity analysis

# Vanishing viscosity analysis

goal: solve differential inclusion for $\varepsilon = 0$ by passing to the limit of the viscous solutions $z_\varepsilon$

problem:
1. a priori estimates in non-reflexive spaces
2. limit element is not differentiable in general

strategy:
1. rescale solutions such that rescaled solutions $\hat{z}_\varepsilon$ are uniformly equicontinuous, use Arzelà-Ascoli and obtain a limit $\hat{z}$
2. introduce generalized notion of differentiability fulfilled by $\hat{z}$

# Vanishing viscosity analysis: time parameterization

recall the energy dissipation balance:

$$\mathcal{I}(\ell(s), z_\varepsilon(s)) - \mathcal{I}(\ell(t), z_\varepsilon(t)) + \int_s^t \partial_r \mathcal{I}(\ell(r), z_\varepsilon(r))\,\mathrm{d}r$$

$$= \int_s^t \mathcal{R}(\dot{z}_\varepsilon(r)) + \underbrace{\tfrac{\varepsilon}{2}\|\dot{z}_\varepsilon(s)\|_{\mathcal{V}}^2 + \tfrac{1}{2\varepsilon}\mathrm{dist}_{\mathcal{V}^*}(-D_z\mathcal{I}(\ell(r), z_\varepsilon(r)), \partial\mathcal{R}(0))^2}_{\xrightarrow{\varepsilon\to 0}?}\,\mathrm{d}r$$

# Vanishing viscosity analysis: time parameterization

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set
$$\begin{cases} \mathfrak{p}(v,w) & := \inf_{\varepsilon>0}\Big(\mathcal{R}(v) + \tfrac{\varepsilon}{2}\|v\|_{\mathcal{V}}^2 + \tfrac{1}{2\varepsilon}\mathrm{dist}_{\mathcal{V}^*}(w, \partial\mathcal{R}(0))^2\Big) \\ s_\varepsilon(t) & := t + \int_0^t \mathfrak{p}(\dot{z}_\varepsilon(r), -D_z\mathcal{I}(\ell(r), z_\varepsilon(r)))\,\mathrm{d}r\,, \\ S_\varepsilon & := s_\varepsilon(T). \end{cases}$$

Since $s_\varepsilon$ is strictly increasing, the inverse $\hat{t}_\varepsilon := s_\varepsilon^{-1} : [0, S_\varepsilon] \to [0, T]$ exists, and we define

$$\hat{z}_\varepsilon : [0, S_\varepsilon] \to \mathcal{Z},\ \hat{z}_\varepsilon(s) := z_\varepsilon(\hat{t}_\varepsilon(s)).$$

# Vanishing viscosity analysis: time parameterization

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$$= \int_s^t \mathcal{R}(\dot{z}_\varepsilon(r)) + \underbrace{\tfrac{\varepsilon}{2}\|\dot{z}_\varepsilon(s)\|_{\mathcal{V}}^2 + \tfrac{1}{2\varepsilon}\mathrm{dist}_{\mathcal{V}^*}(-D_z\mathcal{I}(\ell(r), z_\varepsilon(r)), \partial\mathcal{R}(0))^2}_{\xrightarrow{\varepsilon\to 0}?}\,\mathrm{d}r$$

set
$$\begin{cases} \mathfrak{p}(v,w) & := \mathcal{R}(v) + \|v\|_{\mathcal{V}}\mathrm{dist}_{\mathcal{V}^*}(w, \partial\mathcal{R}(0)) \text{ for } v \in \mathcal{Z}, w \in \mathcal{V}^* \\ s_\varepsilon(t) & := t + \int_0^t \mathfrak{p}(\dot{z}_\varepsilon(r), -D_z\mathcal{I}(\ell(r), z_\varepsilon(r)))\,\mathrm{d}r\,, \\ S_\varepsilon & := s_\varepsilon(T). \end{cases}$$

Since $s_\varepsilon$ is strictly increasing, the inverse $\hat{t}_\varepsilon := s_\varepsilon^{-1} : [0, S_\varepsilon] \to [0, T]$ exists, and we define

$$\hat{z}_\varepsilon : [0, S_\varepsilon] \to \mathcal{Z},\ \hat{z}_\varepsilon(s) := z_\varepsilon(\hat{t}_\varepsilon(s)).$$

# Vanishing viscosity analysis: Arzelà-Ascoli argument

- thanks to the choice of parameterization, it holds almost everywhere

$$\dot{\hat{t}}_\varepsilon(s) + \mathcal{R}(\dot{\hat{z}}_\varepsilon(s)) + \|\dot{\hat{z}}_\varepsilon(s)\|_\mathcal{V} \mathrm{dist}_{\mathcal{V}^*}(-D_z\mathcal{I}(\ell(\hat{t}_\varepsilon(s)), \hat{z}_\varepsilon(s)), \partial\mathcal{R}(0)) = 1$$

  - therefore, it holds that $[0, S] := \bigcup_\varepsilon [0, S_\varepsilon]$ is compact

- Ehrling's lemma yields $W^{1\infty}(0, S; \mathcal{X}) \cap L^\infty(0, S; \mathcal{Z}) \subset C([0, S], \mathcal{V})$
  - thus, $\sup_\varepsilon \mathcal{R}(\dot{\hat{z}}_\varepsilon(s)) \leq 1$ implies uniform equicontinuity of $\hat{z}_\varepsilon$ details

- a priori estimate $\sup_\varepsilon \|z_\varepsilon\|_{L^\infty(0,T;\mathcal{Z})} < \infty$ implies that the set $K := \{\hat{z}_\varepsilon(s) \,|\, \varepsilon > 0,\ s \in [0, S]\}$ is compact w.r.t. $\overset{\mathcal{Z}}{\rightharpoonup}$.

- thus, there is $\hat{z} \in C([0, S], \mathcal{V})$ such that $\hat{z}_{\varepsilon_n}(s) \xrightarrow{n\to\infty} \hat{z}(s)$ in $\mathcal{Z}$

# Vanishing viscosity analysis: $\mathcal{R}$-absolute continuity

## Definition and Proposition

A map $f : [0, T] \to \mathbb{R}$ is called **absolutely continuous** if and only if

$$\exists g \in L^1(0, T) : \forall\, 0 \leq r < s \leq T : \quad |f(s) - f(r)| \leq \int_r^s g(t)\,\mathrm{d}t\,.$$

In this case, for almost all $t \in [0, T]$, the limit

$$f'(t) := \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$$

exists and $|f'|$ is the least function fulfilling the above estimate.

# Vanishing viscosity analysis: $\mathcal{R}$-absolute continuity

## Definition and Proposition

A map $z : [0, T] \to \mathcal{Z}$ is called $\mathcal{R}$-**absolutely continuous** if and only if

$$\exists g \in L^1(0, T) : \forall\, 0 \leq r < s \leq T : \mathcal{R}(z(s) - z(r)) \leq \int_r^s g(t)\,\mathrm{d}t\,.$$

In this case, for almost all $t \in [0, T]$, the limit

$$\mathcal{R}[z'](t) := \lim_{h \searrow 0} \frac{\mathcal{R}(z(t+h) - z(t))}{h} \in \mathbb{R}$$

exists and $\mathcal{R}[z']$ is the least function fulfilling the above estimate.

$\mathcal{R}[z']$ is called the **generalized metric derivative.**

Ambrosio, Gigli, Savaré. **Gradient Flows in Metric Spaces and in the Space of Probability Measures.** 2008.

# Vanishing viscosity analysis: regularity

From the identity

$$\dot{\hat{t}}_\varepsilon(s) + \underbrace{\mathcal{R}(\dot{\hat{z}}_\varepsilon(s))}_{\widehat{=}\mathcal{R}[\hat{z}'_\varepsilon](s)} + \|\dot{\hat{z}}_\varepsilon(s)\|_\mathcal{V} \underbrace{\mathrm{dist}_{\mathcal{V}^*}(-D_z\mathcal{I}(\ell(\hat{t}_\varepsilon(s)), \hat{z}_\varepsilon(s)), \partial\mathcal{R}(0))}_{=:\mathfrak{e}(\hat{t}_\varepsilon(s),\hat{z}_\varepsilon(s))} = 1,$$

we conclude that

- there is $\hat{t} \in W^{1,\infty}(0,S)$ such that $\hat{t}_\varepsilon \overset{*}{\rightharpoonup} \hat{t}$,
- $\hat{z} \in \mathrm{AC}([0,S];\mathcal{R})$ with $\mathcal{R}[\hat{z}'] \leq 1$,
- for $[a,b] \subset \{s \in [0,S] \,|\, \mathfrak{e}(\hat{t}(s),\hat{z}(s)) > 0\}$, it holds $\hat{z} \in \mathrm{AC}([a,b];\mathcal{V})$.

Mielke, Rossi, Savaré. **Balanced viscosity solutions to infinite-dimensional rate-independent systems.** 2016.

# Energy dissipation balance - upper bound

Change of variable in the energy dissipation balance yields

$$\mathcal{I}(\ell(\hat{t}_\varepsilon(0)), \hat{z}_\varepsilon(0)) + \int_0^s \partial_\ell \mathcal{I}(\ell(\hat{t}_\varepsilon(r)), \hat{z}_\varepsilon(r)) \dot{\ell}(\hat{t}_\varepsilon(r)) \dot{\hat{t}}_\varepsilon(r) \,\mathrm{d}r$$

$$= \mathcal{I}(\ell(\hat{t}(s)), \hat{z}_\varepsilon(s)) + \int_0^s \mathcal{R}(\dot{\hat{z}}_\varepsilon(r)) \,\mathrm{d}r$$

$$+ \int_0^s \frac{\varepsilon}{2\dot{\hat{t}}_\varepsilon(r)} \|\dot{\hat{z}}_\varepsilon(r)\|_{\mathcal{V}}^2 + \frac{\dot{\hat{t}}_\varepsilon(r)}{2\varepsilon} \mathrm{dist}_{\mathcal{V}^*}(-D_z \mathcal{I}(\ell(\hat{t}_\varepsilon(r)), \hat{z}_\varepsilon(r)), \partial\mathcal{R}(0))^2 \,\mathrm{d}r \,.$$

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$$\mathcal{I}(\ell(\hat{t}_\varepsilon(0)), \hat{z}_\varepsilon(0)) + \int_0^s \partial_\ell \mathcal{I}(\ell(\hat{t}_\varepsilon(r)), \hat{z}_\varepsilon(r))\dot{\ell}(\hat{t}_\varepsilon(r))\dot{\hat{t}}_\varepsilon(r)\,\mathrm{d}r$$

$$\geq \mathcal{I}(\ell(\hat{t}(s)), \hat{z}_\varepsilon(s)) + \int_0^s \mathcal{R}(\dot{\hat{z}}_\varepsilon(r))\,\mathrm{d}r$$

$$+ \int_0^s \|\dot{\hat{z}}_\varepsilon(r)\|_{\mathcal{V}} \mathrm{dist}_{\mathcal{V}^*}(-D_z\mathcal{I}(\ell(\hat{t}_\varepsilon(r)), \hat{z}_\varepsilon(r)), \partial\mathcal{R}(0))\,\mathrm{d}r\,.$$

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$$+ \int_0^s \|\dot{\hat{z}}_\varepsilon(r)\|_{\mathcal{V}} \mathrm{dist}_{\mathcal{V}^*}(-D_z\mathcal{I}(\ell(\hat{t}_\varepsilon(r)), \hat{z}_\varepsilon(r)), \partial\mathcal{R}(0))\,\mathrm{d}r\,.$$

By lower semicontinuity arguments, we obtain the estimate

$$\mathcal{I}(\ell(\hat{t}(0)),\hat{z}(0)) + \int_0^s \partial_\ell \mathcal{I}(\ell(\hat{t}(r)), \hat{z}(r))\dot{\ell}(\hat{t}(r))\dot{\hat{t}}(r)\,\mathrm{d}r$$

$$\geq \mathcal{I}(\ell(\hat{t}(s)), \hat{z}(s)) + \int_0^s \mathcal{R}[\hat{z}'](r) + \|\dot{\hat{z}}(r)\|_{\mathcal{V}}\mathfrak{e}(\hat{t}(r), \hat{z}(r))\,\mathrm{d}r\,.$$

# Vanishing viscosity solutions - existence

## Definition and Theorem

*There are* $(t, z) \in W^{1,\infty}(0, S) \times AC(0, S; \mathcal{R})$ *fulfilling:*

$$t(S) = T, \quad \|z\|_{L^\infty(0,S;\mathcal{Z})} < \infty,$$

$$\text{f.a.a. } s \in [0, S] : \begin{cases} \dot{t}(s) \geq 0, \quad \dot{t}(s)\mathfrak{e}(t(s), z(s)) = 0, \\ \dot{t}(s) + \mathcal{R}[z'](s) + \|\dot{z}(s)\|_{\mathcal{V}}\mathfrak{e}(t(s), z(s)) = 1, \end{cases}$$

*for* $G := \{s \in [0, S] \,|\, \mathfrak{e}(t(s), z(s)) > 0\}$, *we have* $z \in AC_{loc}(G; \mathcal{V})$,

$$\mathcal{I}(\ell(0), z_0) + \int_0^s \partial_\ell \mathcal{I}(\ell(t(r)), z(r))\dot{\ell}(t(r))\dot{t}(r)\,\mathrm{d}r$$
$$= \mathcal{I}(\ell(t(s)), z(s)) + \int_0^s \mathcal{R}[z'](r) + \|\dot{z}(r)\|_{\mathcal{V}}\mathfrak{e}(t(r), z(r))\,\mathrm{d}r$$

*A triple* $(S, t, z) \in [T, \infty) \times W^{1,\infty}(0, S) \times AC(0, S; \mathcal{R})$ *fulfilling the above conditions is called* **normalized parameterized solution** *of the system* $0 \in \partial\mathcal{R}(\dot{z}(t)) + D_z\mathcal{I}(\ell(t), z(t))$ *a.e. in* $[0, T]$, $\quad z(0) = z_0$.

prove lower bound

# Vanishing viscosity solutions - interpretation

We have the complementarity condition

$$\dot{t}(s)\mathrm{dist}_{\mathcal{V}^*}(-D_z\mathcal{I}(\ell(t(s)), z(s)), \partial\mathcal{R}(0)) = 0.$$

- If $\dot{t}(s) > 0$, then $\mathrm{dist}_{\mathcal{V}^*}(-D_z\mathcal{I}(\ell(t(s)), z(s)), \partial\mathcal{R}(0)) = 0$ and $z(s)$ is a stationary solution of the rate-independent system.
  - $\dot{z}(s) = 0$: sticking
  - $\dot{z}(s) \neq 0$: rate-independent sliding
- If $\mathrm{dist}_{\mathcal{V}^*}(-D_z\mathcal{I}(\ell(t(s)), z(s)), \partial\mathcal{R}(0)) > 0$, then $\dot{t}(s) = 0$, and due to the normalization condition, we have $\dot{z}(s) \neq 0$. Thus, the artificial time is frozen and the state changes according to viscous laws. In the outer time scale, this is seen as a jump.

# Summary

- Rate-independent systems and vanishing viscosity solutions
  - approximate system by adding artificial viscosity
- Solve the viscous system
- Find limit for vanishing viscosity
  - find suitable parameterization
  - introduce metric derivatives
  - existence and regularity of limit element
  - identify energy-dissipation balance
- Next goal: optimal control with control variable $\ell$
  - prove sufficient a priori estimates/compactness results

# Existence of viscous solutions via time discretization

- choose a partition $\{0 = t_0^\tau < t_1^\tau < \cdots < t_{N-1}^\tau < T \leq t_N^\tau\}$ of $[0, T]$ with step length $\tau > 0$
- set $z_0^{\tau,\varepsilon} := z_0$ and obtain $(z_k^{\tau,\varepsilon})_{k=1,\dots,N}$ as solutions of
$$z_{k+1}^{\tau,\varepsilon} \in \operatorname{Argmin}\{\mathcal{I}(\ell(t_{k+1}^\tau), z) + \tau \mathcal{R}_\varepsilon\left(\frac{z - z_k^{\tau,\varepsilon}}{\tau}\right), \ z \in \mathcal{Z}\}.$$
- interpolants $\hat{z}_\varepsilon^\tau, \overline{z}_\varepsilon^\tau$ fulfill the energy dissipation estimate
$$\int_0^T \mathcal{R}_\varepsilon(\dot{\hat{z}}_\varepsilon^\tau(t)) + \mathcal{R}_\varepsilon^*(-D_z\mathcal{I}(\ell(\underline{\boldsymbol{t}}^\tau(t)), \overline{z}_\varepsilon^\tau(t)))\,\mathrm{d}t + \mathcal{I}(\ell(T), \hat{z}_\varepsilon^\tau(T))$$
$$\leq \mathcal{I}(\ell(0), z_0) + \int_0^T \partial_t \mathcal{I}(\ell(t), \hat{z}_\varepsilon^\tau(t))\,\mathrm{d}t + \int_0^T r_\tau(t)\,\mathrm{d}t$$
- this carries over to the limit due to the lower semicontinuity of $\mathcal{R}$, $\mathcal{R}^*$

back to theorem

# Existence of viscous solutions via time discretization

- we have the chain rule

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}(\ell(t), z_\varepsilon(t)) = \langle D_z\mathcal{I}(\ell(t), z_\varepsilon(t)), \dot{z}_\varepsilon(t)\rangle_{\mathcal{Z}} + \partial_t\mathcal{I}(\ell(t), z_\varepsilon(t))$$

- Fenchel-Moreau inequality

$$\mathcal{R}_\varepsilon(\dot{z}_\varepsilon(r)) + \mathcal{R}_\varepsilon^*(-D_z\mathcal{I}(\ell(r), z_\varepsilon(r))) \geq \langle -D_z\mathcal{I}(\ell(t), z_\varepsilon(t)), \dot{z}_\varepsilon(t)\rangle_{\mathcal{Z}}$$

and integration w.r.t. time then yield the opposite estimate

$$\int_s^t \mathcal{R}_\varepsilon(\dot{z}_\varepsilon(r)) + \mathcal{R}_\varepsilon^*(-D_z\mathcal{I}(\ell(r), z_\varepsilon(r)))\,\mathrm{d}r + \mathcal{I}(\ell(t), z_\varepsilon(t)$$
$$\geq \mathcal{I}(\ell(s), z_\varepsilon(s)) + \int_s^t \partial_t\mathcal{I}(\ell(r), z_\varepsilon(r))\,\mathrm{d}r$$

back to theorem

# Existence of viscous solutions via time discretization

- we have the chain rule

$$\int_s^t \langle -D_z\mathcal{I}(\ell(r), z_\varepsilon(r)), \dot{z}_\varepsilon(r)\rangle_{\mathcal{Z}} \,\mathrm{d}r + \mathcal{I}(\ell(t), z_\varepsilon(t)) = \mathcal{I}(\ell(s), z_\varepsilon(s)) + \int_s^t \partial_t\mathcal{I}(\ell(r), z_\varepsilon(r))\,\mathrm{d}r$$

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$$\mathcal{R}_\varepsilon(\dot{z}_\varepsilon(r)) + \mathcal{R}_\varepsilon^*(-D_z\mathcal{I}(\ell(r), z_\varepsilon(r))) \geq \langle -D_z\mathcal{I}(\ell(t), z_\varepsilon(t)), \dot{z}_\varepsilon(t)\rangle_{\mathcal{Z}}$$

and integration w.r.t. time then yield the opposite estimate

$$\int_s^t \mathcal{R}_\varepsilon(\dot{z}_\varepsilon(r)) + \mathcal{R}_\varepsilon^*(-D_z\mathcal{I}(\ell(r), z_\varepsilon(r)))\,\mathrm{d}r + \mathcal{I}(\ell(t), z_\varepsilon(t)) \geq \mathcal{I}(\ell(s), z_\varepsilon(s)) + \int_s^t \partial_t\mathcal{I}(\ell(r), z_\varepsilon(r))\,\mathrm{d}r$$

back to theorem

# Equicontinuity via Ehrling's Lemma

## Lemma (Mielke, Rossi, Savaré 2016)

*For bounded $K \subset \mathcal{Z}$, there is an increasing, continuous from the right function $\Omega_K : [0,\infty) \to [0,\infty)$ such that $\Omega_K(0) = 0$ and for all $z_1, z_2 \in K$, it holds:*

$$\|z_1 - z_2\|_{\mathcal{V}} \leq \Omega_K(\mathcal{R}(z_1 - z_2)).$$

## Proof.

- Ehrling's interpolation lemma: if $\mathcal{Z} \Subset \mathcal{V} \subset \mathcal{X}$, then for all $\eta > 0$, there is $C_\eta > 0$:
$$\forall\, z \in \mathcal{Z} : \quad \|z\|_{\mathcal{V}} \leq \eta \|z\|_{\mathcal{Z}} + C_\eta \|z\|_{\mathcal{X}}$$
- since $K$ is bounded and $\|\cdot\|_{\mathcal{X}} \leq C\mathcal{R}(\cdot)$, this implies: for all $\eta > 0$, there is $C_\eta > 0$:
$$\forall\, z \in K : \quad \|z\|_{\mathcal{V}} \leq \eta + C_\eta \mathcal{R}(z)$$
- set $\Omega_K(r) := \inf_{\eta > 0} \{\eta + C_\eta r\}$ for $r \geq 0$

□

go back

# Equicontinuity via Ehrling's Lemma

- a priori estimate $\sup_\varepsilon \|z_\varepsilon\|_{\mathrm{L}^\infty(0,T;\mathcal{Z})} < \infty$ implies that the set $K := \{\hat{z}_\varepsilon(s) \,|\, \varepsilon > 0,\ s \in [0,S]\}$ is bounded in $\mathcal{Z}$
- thus, for $0 \leq r < s \leq S$, $\varepsilon > 0$:

$$\begin{aligned}\|\hat{z}_\varepsilon(s) - \hat{z}_\varepsilon(r)\|_{\mathcal{V}} &\leq \Omega_K(\mathcal{R}(\hat{z}_\varepsilon(s) - \hat{z}_\varepsilon(r))) \\ &\leq \Omega_K\Big(\int_r^s \underbrace{\mathcal{R}(\dot{\hat{z}}_\varepsilon(\sigma))}_{\leq 1}\,\mathrm{d}\sigma\Big) \leq \Omega_K(s-r)\end{aligned}$$

go back

# Energy dissipation balance - lower bound

For curves $(t, z)$ of the given regularity, $s \mapsto \mathcal{I}(\ell(t(s)), z(s))$ is absolutely continuous and it holds almost everwhere

$$|\tfrac{\mathrm{d}}{\mathrm{d}s}\mathcal{I}(\ell(t(s)), z(s)) - \partial_t \mathcal{I}(\ell(t(s)),z(s))\dot{t}(s)| \\ \leq \mathcal{R}[z'](s) + \|\dot{z}(s)\|_{\mathcal{V}}\mathfrak{e}(t(s), z(s)).$$

Thus, integration w.r.t. $s$ yields the opposite estimate and hence the following identity

$$\mathcal{I}(\ell(\hat{t}(0)),\hat{z}(0)) + \int_0^s \partial_\ell \mathcal{I}(\ell(\hat{t}(r)), \hat{z}(r))\dot{\ell}(\hat{t}(r))\dot{\hat{t}}(r)\,\mathrm{d}r \\ = \mathcal{I}(\ell(\hat{t}(s)), \hat{z}(s)) + \int_0^s \mathcal{R}[\hat{z}'](r) + \|\dot{\hat{z}}(r)\|_{\mathcal{V}}\mathfrak{e}(\hat{t}(r), \hat{z}(r))\,\mathrm{d}r\,.$$

back to definition

# Vanishing viscosity solutions - a priori bounds

**Lemma**

*For all $L > 0$, there is a constant $C_L > 0$ such that for all $z_0 \in \mathcal{Z}$, $\ell \in W^{1,\infty}(0,T;\mathcal{V}^*)$ fulfilling $\|z_0\|_{\mathcal{Z}} + \|\ell\|_{H^1(0,T;\mathcal{V}^*)} \leq L$ and all normalized parameterized solutions $(S, t, z)$ associated with the initial value $z_0$ and the external load $\ell$, it holds:*

$$S + \|z\|_{L^\infty(0,S;\mathcal{Z})} + \mathcal{I}(\ell(t(S)), z(S)) < C_L.$$

# Literature


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