Open and solved problems in the stability of mechanical, electromechanical and electronic systems
(plenary lecture)

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Content

- Linear stationary stabilization: Nyquist criterion (1932)
- Linear nonstationary stabilization:
  - Brockett’s stabilization problem (2000)
  - low frequency stabilization of crane-container and helicopter-container
- Nonlinear stabilization
  - self-excited and hidden oscillations (2010)
  - counterexamples to Aizerman’s (1949) and Kalman’s (1957) conjectures on absolute stability
  - hidden oscillations in drilling systems (2009)
  - hidden oscillations in aircraft control system (1992)
  - hidden attractors in Chua’s circuits (2010)
I. Linear stabilization

Nyquist criterion and various generalizations
Vibration stabilization and control $K = K(t)$

Stabilization of pendulum upper equilibrium vertically vibrating point of suspension

- S.Meerkov 1973
- R.Bellman, J.Bentsman, S.Meerkov 1986
- A.Stephenson 1907, P. Kapitsa 1951, V.Chelomey 1956
- I.Blekhman 2006

$asin\omega t \quad \omega \gg 1$
Brockett stabilization problem

Problem: R. Brockett (MIT; IEEE and AMS Fellow) *Open problems in mathematical systems and control theory*, Springer, 2000

The problem is to define $K(t)$ for given matrixes $A, B, C$ such that

$$\frac{dx}{dt} = (A + BK(t)C)x$$

is asymptotically stable?

The problem is to define $K(t)$ for given matrixes $A, B, C$ such that
\[ \frac{dx}{dt} = (A + BK(t)C)x \]
is asymptotically stable?

In other words: *how the nonstationary stabilization extends the possibilities of stabilization?*

Two approaches for stabilization:
- high frequencies: $K(t) = a\sin\omega t$, $\omega \gg 1$
  L.Moreau, D.Aeyels 2004
- low frequencies: $K(t) = a\text{sign}\sin\omega t$, $\omega \ll 1$
For two dimensional system — necessary and sufficient conditions of stabilization:
Transfer function

\[ W(p) = \frac{p + c}{p^2 + b_2p + b_1} \quad (B \text{ and } C - \text{vectors}) \]

Non singularity: \( c^2 - b_2c + b_1 \neq 0 \)
Stationary stabilization \( K(t) \equiv K_0: \) \( c > 0, \) or \( c \leq 0, \quad a_2c < a_1 \)
Let \( c \leq 0, \quad a_2c \geq a_1 \) (stationary stabilization is impossible)

For nonstationary stabilization necessary and sufficient

\[ c^2 - a_2c + a_1 > 0. \]

Discrete analog

\[ x(k + 1) = (A + BK(k)C)x(k) \]

Two-dimensional system can be stabilize by low frequency for almost all parameters \((CA^{-1}B \neq 0, \quad W(p) - \text{nonsingular})\).
Low frequency stabilization of crane.

Control of a Parametrically Excited Crane, J.Collado, C.Vazquez, Mexico 2010
Low frequency stabilization of helicopter-container
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  - hidden oscillations in phase locked loops (1961)
II. Nonlinear theory

\[ z = f(y) \]

Linear object

\[ y \]
Hidden oscillations (3d): Aizerman and Kalman conjectures

If \( \dot{z} = Az + bk c^*z \), is asympt. stable \( \forall k \in (k_1, k_2) : \forall z(t, z_0) \to 0 \), then is \( \dot{x} = Ax + b \varphi(\sigma), \sigma = c^* x, \varphi(0) = 0, k_1 < \varphi(\sigma)/\sigma < k_2 : \forall x(t, x_0) \to 0 \)?

1949 : \( k_1 < \varphi(\sigma)/\sigma < k_2 \)
1957 : \( k_1 < \varphi'(\sigma) < k_2 \)

In general, conjectures are not true (Aizerman’s: \( n \geq 2 \), Kalman’s: \( n \geq 4 \))

Periodic solution can exist for nonlinearity from linear stability sector

Aizerman’s conj.: N. Krasovsky 1952: \( n = 2, x(t) \to \infty \); V. Pliss 1958: \( n = 3 \), periodic \( x(t) \)

Engineering analytical-numerical analysis of dynamical system

\[ \dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{pmatrix} = \begin{pmatrix} f_1(x_1, x_2, \ldots, x_n) \\ f_2(x_1, x_2, \ldots, x_n) \\ \vdots \\ f_n(x_1, x_2, \ldots, x_n) \end{pmatrix} = F(x) \]

- 1. Find equilibria
  \[ x^k : F(x^k) = 0 \]
  local stability analysis
  \[ \dot{y} = \frac{dF(x)}{dx} \bigg|_{x=x^k} y \]

Computation of oscillations and attractors

**self-excited attractor localization:** *standard computational procedure* is 1) to find equilibria; 2) after transient process trajectory, starting from a point of unstable manifold in a neighborhood of unstable equilibrium, reaches an self-excited oscillation and localizes it.

**Van der Pol**
\[
\begin{align*}
&\dot{x} = y \\
&\dot{y} = -x + \varepsilon(1-x^2)y
\end{align*}
\]

**Lorenz**
\[
\begin{align*}
&x = -\sigma(x - y) \\
y = rx - y - xz \\
z = -bz + xy
\end{align*}
\]

**hidden attractor:** basin of attraction does not intersect with a small neighborhood of equilibria [Leonov, Kuznetsov, Vagaitsev, Phys. Lett. A, 375, 2011, 2230-2233]

- **✓ standard computational procedure** does not work: all equilibria are stable or not in the basin of attraction
- **✓ integration with random initial data** does not work: basin of attraction is small, system’s dimension is large

How to choose initial data in the attraction domain?
Nonexistence of oscillations (sufficient conditions):
Popov criteria (1958) \( 0 < \frac{f(y)}{y} < k \)

\[
\frac{1}{k} + \text{Re} (1 + q i \omega) W(i \omega) > 0 \quad W(p) = C(A - p I)^{-1} B, \forall \omega,
\]

Hidden oscillations:
Pliss counterexample to Aizerman’s conjecture (1956)
Counterexample to Aizerman’s conjecture with typical engineering nonlinearity

\[ \dot{x} = Ax + b \varphi(\sigma), \quad \sigma = c^*x, \quad \varphi(0) = 0, \quad 0 < \varphi(\sigma) < k : \quad \forall x(t, x_0) \to 0? \]

Typical nonlinearity \( f(y) = \text{sat}(y) \)

- Bernat J. & Llibre J. 1996
- Leonov G.A., Kuznetsov N.V., Bragin V.O. 2010
Kalman conjecture (Kalman problem) 1957

\[ \dot{x} = Ax + b \varphi(\sigma), \quad \sigma = c^*x, \quad \varphi(0) = 0, \quad 0 < \varphi'(\sigma) < k : \quad \forall x(t, x_0) \to 0? \]

- **Fitts R. 1966**: series of counterexamples in \( \mathbb{R}^4 \), nonlinearity \( \varphi(\sigma) = \sigma^3 \)
- **Barabanov N. 1979-1988**: some of Fitts counterexamples are false; analytical ‘counterex.’ construction in \( \mathbb{R}^4 \), \( \varphi(\sigma) \) ‘close’ to \( \text{sign}(\sigma) (0 \leq \varphi'(\sigma)) \)
  - *later ‘gaps’ were reported by Glutsyuk, Meisters, Bernat & Llibre*
- **Leonov G. et al. 1996**: Kalman conj. is true in \( \mathbb{R}^3 \) (by YKP freq. methods)
- **Bernat J. & Llibre J. 1996**: analytical-numerical ‘counterexamples’ construction in \( \mathbb{R}^4 \), \( \varphi(\sigma) \) ‘close’ to \( \text{sat}(\sigma) (0 \leq \varphi'(\sigma)) \)
- **Leonov G., Kuznetsov N., Bragin V. 2010**: some of Fitts counterexamples are true; smooth counterexample in \( \mathbb{R}^4 \) with \( \varphi(\sigma) = \tanh(\sigma) \): \( 0 < \tanh'(\sigma) \leq 1 \);
  - analytical-numerical counterexamples construction for any type of \( \varphi(\sigma) \);

**Hidden oscillation in aircraft control systems**

**Windup** – oscillations with increasing amplitude
- Crash - YF-22 Raptor (Boeing) 1992
- Crash - JAS-39 Gripen (SAAB) 1993

**Antiwindup** – an additional scheme to avoid windup effect in system with saturation

Lauvdal, Murray, Fossen, Stabilization of integrator chains in the presence of magnitude and rate saturations; a gain scheduling approach, *Proc. of CDC, 1997*: “Since stability in simulations does not imply stability of the physical control system (an example is the crash of the YF22) stronger theoretical understanding is required”

The angle-of-attack control system of the unstable aircraft model in the presence of saturation in magnitude of the control input: stable zero equilibrium coexist with undesired stable oscillation — hidden oscillations

PID controller:

\[ u = k_I \int_0^t e(\tau) d\tau + k_P e + k_D q \]

\[ e = \alpha - \alpha^* \]

Aircraft angular motion:

\[ \dot{\alpha} = Z_\alpha \alpha + q + Z_\delta \delta_e \]
\[ \ddot{q} = M_\alpha \alpha + M_q q + M_\delta \delta_e \]

Elevator servosystem & look-ahead compensator:

\[ u = \frac{k(T_2^2 + 2\xi_2 T_2 s + 1)}{T_1^2 + 2\xi_1 T_1 s + 1} \]

Saturated elevator:

Hidden oscillation: Drilling system

Drill string failure ≈ 1 out of 7 drilling rigs, costs $100 000 (www.oilgasprod.com)

De Bruin, J.C.A. et al. (2009), *Automatica*, 45(2), 405–415

\[
\begin{align*}
\dot{\omega}_u &= -k_\theta \alpha - T_{fu}(\omega_u) + k_m u, \\
\dot{\omega}_l &= k_\theta \alpha - T_{fl}(\omega_l), \\
\dot{\alpha} &= \omega_u - \omega_l, \\
\omega_{u,l} &= \dot{\theta}_{u,l}, \quad \alpha = \theta_u - \theta_l, \\
T_{fu}, T_{fl} &- friction torque
\end{align*}
\]

Operating mode: angular displacement between upper and lower discs \(\alpha = \theta_u - \theta_l \rightarrow \text{const.}\)

Hidden oscillation: stable limit cycle coexists with stable equilibrium

Hidden oscillation of \((\theta_u - \theta_l)\):
- is difficult to find by standard simulation
- may lead to breakdown
Hidden oscillation: Drilling system with induction motor

Hidden oscillation: stable limit cycle coexists with stable equilibrium

Hidden oscillation of $(\theta_u - \theta_l)$:
— is difficult to find by standard simulation;
— may lead to breakdown

\[
\begin{align*}
\dot{y} &= -cy - s - xs, \quad \dot{x} = -cx + ys, \\
\dot{\theta} &= u - s, \quad s = -\dot{\theta}_u, \quad u = -\dot{\theta}_l \\
\dot{s} &= \frac{k_{\theta}}{J_u} \theta + \frac{b}{J_u} (u - s) + \frac{a}{J_u} y, \\
\dot{u} &= -\frac{k_{\theta}}{J_l} - \frac{b}{J_l} (u - s) + \frac{1}{J_l} T_{fl} (\omega - u),
\end{align*}
\]


Attractors in Chua’s circuits

\[
\begin{align*}
\dot{x} &= \alpha(y - x - f(x)), \\
\dot{y} &= x - y + z, \\
\dot{z} &= -(\beta y + \gamma z), \\
f(x) &= m_1 x + \text{sat}(x) = m_1 x + \frac{1}{2} (m_0 - m_1) (|x + 1| + |x - 1|)
\end{align*}
\]

Chua circuit is used in chaotic communications


[\text{Bilotta&Pantano, A gallery of Chua attractors, WorldSci. 2008}]

Could an attractor exists and how lo localize it if equilibrium is stable?

L.Chua, 1990: If zero equilibrium is stable \(\Rightarrow\) no chaotic attractor
In 2010 the notion of hidden attractor was introduced and hidden chaotic attractor was found for the first time:

\[ \dot{x} = \alpha(y - x - m_1 x - \psi(x)) \]
\[ \dot{y} = x - y + z, \quad \dot{z} = - (\beta y + \gamma z) \]
\[ \psi(x) = (m_0 - m_1) \text{sat}(x) \]

\[ \alpha = 8.4562, \quad \beta = 12.0732 \]
\[ \gamma = 0.0052 \]
\[ m_0 = -0.1768, \quad m_1 = -1.1468 \]

equilibria: stable zero \( F_0 \) & 2 saddles \( S_{1,2} \)
trajectories: 'from' \( S_{1,2} \) tend (black) to zero \( F_0 \) or tend (red) to infinity;
Hidden chaotic attractor (in green) with positive Lyapunov exponent
Lyapunov exponent: chaos, stability, Perron effects, linearization

\[
\begin{align*}
\dot{x} &= F(x), \quad x \in \mathbb{R}^n, \quad F(x_0) = 0 \\
x(t) &\equiv x_0, \quad A = \frac{dF(x)}{dx} \bigg|_{x=x_0}
\end{align*}
\]

\[
\begin{align*}
\dot{y} &= Ay + o(y) \\
y(t) &\equiv 0, \quad (y = x - x_0)
\end{align*}
\]

\[
\begin{align*}
\dot{z} &= Az \\
z(t) &\equiv 0
\end{align*}
\]

✓ stationary: \( z(t) = 0 \) is exp. stable \( \Rightarrow \) \( y(t) = 0 \) is asympt. stable

\[
\begin{align*}
\dot{x} &= F(x), \quad \dot{x}(t) = F(x(t)) \neq 0 \\
x(t) &\neq x_0, \quad A(t) = \frac{dF(x)}{dx} \bigg|_{x=x(t)}
\end{align*}
\]

\[
\begin{align*}
\dot{y} &= A(t)y + o(y) \\
y(t) &\equiv 0, \quad (y = x - x(t))
\end{align*}
\]

? nonstationary: \( z(t) = 0 \) is exp. stable \( \Rightarrow ? \) \( y(t) = 0 \) is asympt. stable

! Perron effect: \( z(t) = 0 \) is exp. stable (unst), \( y(t) = 0 \) is exp. unstable (st)

Positive largest Lyapunov exponent doesn’t, in general, indicate chaos

Further examples of hidden attractors

J.C. Sprott, A conservative dynamical system with a strange attractor, Physics Letters A, 2014
Wei Z., A new finding of the existence of hyperchaotic attractors with no equilibria, Mathematics and Computers in Simulation, 100, 2014
C. Li, J.C. Sprott, Coexisting hidden attractors in a 4-d simplified Lorenz system, IJBC, 2014
Z. Wei, Hidden hyperchaotic attractors in a modified Lorenz-Stenflo system with only one stable equilibrium, Int. J. of Bif.&Chaos, 2014
Q. Li, H. Zeng, X.-S. Yang, On hidden twin attractors and bifurcation in the Chua’s circuit, Nonlinear Dynamics, 2014
H. Zhao, Y. Lin, Y. Dai, Hidden attractors and dynamics of a general autonomous van der Pol-Duffing oscillator, Int. J. of Bif.&Chaos, 2014
Jafari S., Sprott J., Simple chaotic flows with a line equilibrium, Chaos, Sol & Fra, 57, 2013
C. Li, J.C. Sprott, Multistability in a butterfly flow, Int. J. of Bif.&Chaos, 23(12), 2013

Other our papers on hidden attractors

✓ G.A. Leonov, V.I. Vagaitsev, N.V. Kuznetsov, Algorithm for localizing Chua attractors based on the harmonic linearization method, *Doklady Mathematics*, 82(1), 2010, 663-666
Phase-locked loops (PLL): history

- Radio and TV
  - de Bellescize H., La réception synchrone, L’Onde Électrique, 11, 1932

- Computer architectures (frequency multiplication)
  - Ian Young, PLL in a microprocessor i486DX2-50 (1992)
    (in Turbo regime stable operation was not guaranteed)

- Theory and Technology
  - F.M. Gardner, Phase-Lock Techniques, 1966
  - A.J. Viterbi, Principles of Coherent Comm., 1966
  - W.C. Lindsey, Synch. Syst. in Comm. and C., 1972
  - W.F. Egan, Freq. Synthesis by Phase Lock, 2000
  - B. Razavi, Phase-Locking in High-Perf. Syst., 2003
  - R. Best, PLL: Design, Simulation and Appl., 2003
PLL analysis & design

PLL operation: generation of electrical signal (voltage), phase of which is automatically tuned to phase of input reference signal (after transient process the signal, controlling the frequency of tunable osc., is constant)

\[ 0.5(\sin(\omega t + \theta(t)) + \sin(\omega t - \theta(t))) \cos(\theta(t)) \]

Filter \hspace{1cm} VCO \hspace{1cm} 1 \hspace{1cm} 2 \hspace{1cm} \text{REF} \hspace{1cm} g(t)

Design: Signals class (sinusoidal, impulse...), PLL type (PLL, ADPLL, DPLL...)

Analysis: Choose PLL parameters (VCO, PD, Filter etc.) to achieve stable operation for the desired range of frequencies and transient time

Analysis methods: simulation, linear analysis, nonlinear analysis of mathematical models in signal space and phase-frequency space

Leonov, Kuznetsov, Seledzhi
Open and solved problems: stability and stabilization
MMOM Jyu 2014 28/33
D. Abramovitch, ACC-2008 plenary lecture: Full simulation of PLL in signal/time space is very difficult since one has to observe simultaneously very fast time scale of input signals and slow time scale of signal’s phase. How to construct model of signal’s phases?

Simulation in phase space: Could stable operation be guaranteed for all possible inputs, internal blocks states by simulation? N. Gubar’ (1961), hidden oscillation in PLL

\[
\dot{\eta} = \alpha \eta - (1 - a\alpha)(\sin(\sigma) - \gamma), \\
\dot{\sigma} = \eta - a(\sin(\sigma) - \gamma)
\]

Global stability in simulation, but only bounded region of stability in reality.

Leonov, Kuznetsov, Seledzhi Open and solved problems: stability and stabilization
Classical PLL: models in time & phase-freq. domains

?) Control signals in signals and phase-frequency spaces are equal.
?) Qualitative behaviors of signals and phase-freq. models are the same.

\[ \dot{\theta}_j \geq \omega_{\text{min}}, \quad |\dot{\theta}_1 - \dot{\theta}_2| \leq \Delta \omega, \quad |\dot{\theta}_j(t) - \dot{\theta}_j(\tau)| \leq \Delta \Omega, \quad |\tau - t| \leq \delta, \quad \forall \tau, t \in [0, T] \quad (*) \]

**waveforms:**
\[ f^p(\theta) = \sum_{i=1}^{\infty} \left( a_i^p \sin(i\theta) + b_i^p \cos(i\theta) \right), \quad p = 1, 2 \]
\[ b_i^p = \frac{1}{\pi} \int_{-\pi}^{\pi} f^p(x) \sin(ix)dx, \quad a_i^p = \frac{1}{\pi} \int_{-\pi}^{\pi} f^p(x) \cos(ix)dx, \]

**Thm.** If (*)
\[ \varphi(\theta) = \frac{1}{2} \sum_{l=1}^{\infty} \left( (a_i^1 a_i^2 + b_i^1 b_i^2) \cos(l\theta) + (a_i^1 b_i^2 - b_i^1 a_i^2) \sin(l\theta) \right) \]
then \[ |G(t) - g(t)| = O(\delta), \quad \forall t \in [0, T] \]

Costas loop: digital signal demodulation


- signal demodulation in digital communication

\[ m(t) = \pm 1 \text{ — data,} \]
\[ m(t) \sin(2\omega t) \text{ and } m(t) \cos(2\omega t) \text{ can be filtered out,} \]
\[ m(t) \cos(0) = m(t), \]
\[ m(t) \sin(0) = 0 \]

- wireless receivers
- Global Positioning System (GPS)
Questions