Iterative Refinement for Ill-Conditioned Linear Systems

Shin'ichi Oishi[1,2] , Takeshi Ogita[2,1] and Siegfried M. Rump[3]

[1] Department of Computer Science, Faculty of Science and Engineering,
Waseda University, 3-4-1 Okubo, Tokyo 169-8555 Japan
oishi@waseda.jp

[2] CREST, JST

[3] Institute for Reliable Computing, Hamburg University of Technology,
Schwarzenbergstr. 95 21071 Hamburg, Germany
{rump@tu-harburg.de}

1 Introduction

In this paper, we will consider the convergence of iterative refinement for a linear equation

$$Av = b, \tag{1}$$

where $A \in \mathbb{F}^{n\times n}$ and $b \in \mathbb{F}^n$. Here, $\mathbb{F}$ is a set of floating point numbers. Let u be the unit round-off of the working precision and $\kappa(A) = \|A\|_\infty \|A^{-1}\|_\infty$ be the condition number of the problem. For well posed problems, *i.e.*, in case of $u\kappa(A) < 1$, it has been shown [1]-[4] that the iterative refinement improves the forward and backward errors of computed solutions provided that the residuals are evaluated by extended precision, in which the unit round off $\overline{u}$ is the order of u^2, before rounding back to the working precision. In this talk, we will treat ill-conditioned problems with

$$1 < u\kappa(A) < \infty. \tag{2}$$

We can assume without loss of generality that for a certain positive integer k the following is satisfied:

$$u^k\kappa(A) \leqq \beta < 1. \tag{3}$$

In Ref. [7], Rump has shown that for arbitrary ill-conditioned matrices A, we can have good approximate inverses $R_{1:k}$ satisfying $\|R_{1:k}A - I\|_\infty \leqq \alpha < 1$. Here, $R_{1:k}$ is obtained as

$$R_{1:k} = R_1 + R_2 + \cdots + R_k \tag{4}$$

with $R_i \in \mathbb{F}^{n\times n}$. I is the n-dimensional unit matrix. In Ref. [5], we have partially clarified the mechanism of the convergence of Rump's method.

Let $A, B, C \in \mathbb{F}^{n\times n}$. We assume that we have an accurate matrix product calculation algorithm

$$D_{1:k} = [AB - C]_k \tag{5}$$

satisfying

$$\left\| \sum_{i=1}^{k} D_i - (AB - C) \right\|_\infty \leqq cu^k \|AB - C\|_\infty. \tag{6}$$

Such algorithms have been proposed, for instance, by the authors [6] and [8].

Now we propose the following iterative refinement algorithm:

$$v' = [v - R_{1:k}[Av - b]_k]_1. \tag{7}$$

This scheme is a modification of the iterative refinement algorithm proposed in [9]. Put $r_k = [Av - b]_k$ and let $\Phi(v) = [v - R_{1:k}r_k]_1$. Then, we can write

$$v' = \Phi(v). \tag{8}$$

The following holds:

$$v' = v - R_{1:k}[(Av - b) + e_r] + e_m, \tag{9}$$

where $e_r = r_k - (Av - b)$ and $e_m \in \mathbb{R}^n$ satisfying

$$\|e_r\|_\infty \leqq cu^k \|Av - b\|_\infty, \quad \|e_m\|_\infty \leqq cu\|v - R_{1:k} r_k\|_\infty. \tag{10}$$

In this paper, we will show the forward stability and backward stability of this iterative algorithm.

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