CONVERGENCE PROPERTIES OF COMBINED RELAXATION METHODS

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1 Variational Inequalities with Continuous Mappings

We consider the main idea of combined relaxation (CR) methods and implementable algorithms for solving variational inequality problems with continuous single-valued mappings.

1.1 Problem Formulation

Let U be a nonempty, closed and convex subset of the n-dimensional Euclidean space R^n, $G : U \to R^n$ a continuous mapping. The *variational inequality problem* (VI) is the problem of finding a point $u^* \in U$ such that

$$\langle G(u^*), u - u^* \rangle \geq 0 \quad \forall u \in U. \tag{1}$$

It is well known that the solution of VI (1) is closely related with that of the following problem of finding $u^* \in U$ such that

$$\langle G(u), u - u^* \rangle \geq 0 \quad \forall u \in U. \tag{2}$$

Problem (2) may be termed as the dual formulation of VI (DVI). We will denote by U^* (respectively, by U^d) the solution set of problem (1) (respectively, problem (2)).

To obtain relationships between U^* and U^d, we need additional monotonicity type properties of G.

Definition 1.1 Let W and V be convex sets in R^n, $W \subseteq V$, and let $Q : V \to R^n$ be a mapping. The mapping Q is said to be

(a) *strongly monotone* on W if

$$\langle Q(u) - Q(v), u - v \rangle \geq \tau \|u - v\|^2 \qquad \forall u, v \in W;$$

(b) *strictly monotone* on W if

$$\langle Q(u) - Q(v), u - v \rangle > 0 \qquad \forall u, v \in W, u \neq v;$$

(c) *monotone* on W if

$$\langle Q(u) - Q(v), u - v \rangle \geq 0 \qquad \forall u, v \in W;$$

(d) *pseudomonotone* on W if

$$\langle Q(v), u - v \rangle \geq 0 \quad \Longrightarrow \quad \langle Q(u), u - v \rangle \geq 0 \qquad \forall u, v \in W;$$

(e) *quasimonotone* on W if f

$$\langle Q(v), u - v\rangle > 0 \quad \Longrightarrow \quad \langle Q(u), u - v\rangle \geq 0 \qquad \forall u, v \in W;$$

(f) *explicitly quasimonotone* on W if it is quasimonotone on W and

$$\langle Q(v), u - v\rangle > 0 \quad \Longrightarrow \quad \langle Q(z), u - v\rangle > 0 \quad \exists z \in (0.5(u + v), u).$$

It follows from the definitions that the following implications hold:

$$(a) \Longrightarrow (b) \Longrightarrow (c) \Longrightarrow (d) \Longrightarrow (f) \Longrightarrow (e).$$

The reverse assertions are not true in general.

Now we recall the relationships between solution sets of VI and DVI, which are known as the Minty Lemma.

Proposition 1.1 *(i) U^d is convex and closed.*
(ii) $U^d \subseteq U^*$.
(iii) If G is pseudomonotone, $U^* \subseteq U^d$.

The existence of solutions of DVI plays a crucial role in constructing relatively simple solution methods for VI. We note that Proposition 1.1 (iii) does not hold in the (explicitly) quasimonotone case. Moreover, problem (2) may even have no solutions in the quasimonotone case. However, we can give an example of solvable DVI (2) with the underlying mapping G which is not quasimonotone. Nevertheless, as it was shown by Konnov [9], explicitly quasimonotone DVI is solvable under the usual assumptions.

1.2 Classical Iterations

One of most popular approaches for solving general problems of Nonlinear Analysis consists of creating a sequence $\{u^k\}$ such that each u^{k+1} is a solution of some auxiliary problem, which can be viewed as an approximation of the initial problem at the previous point u^k.

At the beginning we consider the simplest problem of solving the nonlinear equation

$$\phi(t) = 0, \tag{3}$$

where $\phi : R \to R$ is continuously differentiable. Recall that the Newton method being applied to this problem iteratively solves the linearized problem

$$\phi(t_k) + \phi'(t_k)(t - t_k) = 0, \tag{4}$$

where t_k is a current iteration point. Obviously, we obtain the well-known process

$$t_{k+1} := t_k - \phi(t_k)/\phi'(t_k), \tag{5}$$

which, under certain assumptions, converges quadratically to a solution of (3). The Newton method for VI (1):

$$\langle G(u^k) + \nabla G(u^k)(u^{k+1} - u^k), v - u^{k+1}\rangle \geq 0 \quad \forall v \in U \tag{6}$$

possesses the same properties. Most modifications of the Newton method consist of replacing the Jacobian $\nabla G(u^k)$ with a matrix A_k. One can then obtain various Newton–like methods such as quasi–Newton methods, successive overrelaxation methods, etc. In general, we can replace (6) with the problem of finding a point $\bar{z} \in U$ such that

$$\langle G(u^k) + \lambda_k^{-1} T_k(u^k, u^{k+1}), v - u^{k+1}\rangle \geq 0 \quad \forall v \in U, \tag{7}$$

where the family of mappings $\{T_k : V \times V \to R^n\}$ such that, for each $k = 0, 1, \ldots,$

(A1) $T_k(u, \cdot)$ *is strongly monotone with constant* $\tau' > 0$ *for every* $u \in U$*;*

(A2) $T_k(u, \cdot)$ *is Lipschitz continuous with constant* $\tau'' > 0$ *for every* $u \in V$*;*

(A3) $T_k(u, u) = 0$ *for every* $u \in V$*.*

For instance, we can choose

$$T_k(u, z) = A_k(z - u) \tag{8}$$

where A_k is an $n \times n$ positive definite matrix. The simplest choice $A_k \equiv I$ in (8) leads to the well-known projection method.

However, all these methods require restrictive assumptions either G be strictly monotone or its Jacobian be symmetric for convergence. We will describe a general approach to constructing iterative solution methods, which involves a solution of the auxiliary problem (7) in order to compute iteration parameters in the main process. As a result, we prove convergence of such a process under rather mild assumptions.

1.3 Basic Properties of CR Methods

We first consider another approach to extending the Newton method (5). Suppose $f : R^n \to R$ is a non-negative, continuously differentiable and convex function. Let us consider the problem of finding a point $u^* \in R^n$ such that

$$f(u^*) = 0. \tag{9}$$

Its solution can be found by the following gradient process

$$u^{k+1} := u^k - (f(u^k)/\|\nabla f(u^k)\|^2)\nabla f(u^k). \tag{10}$$

Note that process (10) can be viewed as some extension of (5). Indeed, the next iterate u^{k+1} also solves the linearized problem

$$f(u^k) + \langle \nabla f(u^k), u - u^k \rangle = 0.$$

Process (10) has quite a simple geometric interpretation. Set

$$H_k = \{u \in R^n \mid \langle \nabla f(u^k), u - u^k \rangle = -f(u^k)\}.$$

Note that the hyperplane H_k separates u^k and the solution set of problem (9). It is easy to see that u^{k+1} in (10) is the projection of u^k onto the hyperplane H_k, so that the distance from u^{k+1} to each solution of (9) has to decrease in comparison with that from u^k.

We now consider an extension of this approach to the case of VI (1).

Definition 1.2 Let W be a nonempty closed set in R^n. A mapping $P : R^n \to R^n$ is said to be *feasible and non-expansive* (f.n.e.) with respect to W, if for every $u \in R^n$, we have

$$P(u) \in W, \|P(u) - w\| \leq \|u - w\| \quad \forall w \in W.$$

We denote by $\mathcal{F}(W)$ the class of all f.n.e. mappings with respect to W. We can take the projection mapping $\pi_W(\cdot)$ as $P \in \mathcal{F}(W)$. However, if the definition of the set U includes functional constraints, then the projection onto U cannot be found by a finite procedure. Nevertheless, in that case there exist finite procedures of finding the corresponding point $P(u)$.

Let us consider an iteration sequence $\{u^k\}$ generated in accordance with the following rules:

$$u^{k+1} \;:=\; P_k(\tilde{u}^{k+1}), \tilde{u}^{k+1} := u^k - \gamma\sigma_k g^k, P_k \in \mathcal{F}(U), \gamma \in (0, 2), \tag{11}$$

$$\langle g^k, u^k - u^* \rangle \geq \sigma_k \|g^k\|^2 \geq 0 \qquad \forall u^* \in U^d. \tag{12}$$

It is easy to see that $\tilde{u}^{k+1}$ is the projection of u^k onto the hyperplane

$$H_k(\gamma) = \{v \in R^n \mid \langle g^k, v - u^k \rangle = -\gamma\sigma_k\|g^k\|^2\},$$

and that $H_k(1)$ separates u^k and U^d. Although $H_k(\gamma)$, generally speaking, does not possess this property, the distance from $\tilde{u}^{k+1}$ to each point of U^d cannot increase and the same is true for u^{k+1} since $U^d \subseteq U$. We now give the key property of the process (11), (12), which specifies the above statement.

Lemma 1.1 *Let a point u^{k+1} be chosen by (11), (12). Then we have*

$$\|u^{k+1} - u^*\|^2 \leq \|u^k - u^*\|^2 - \gamma(2-\gamma)(\sigma_k\|g^k\|)^2 \qquad \forall u^* \in U^d. \tag{13}$$

The following assertions follow immediately from (13).

Lemma 1.2 *Let a sequence $\{u^k\}$ be constructed in accordance with the rules (11), (12). Then:*

(i) $\{u^k\}$ is bounded.

(ii) $\sum_{k=0}^{\infty} \gamma(2-\gamma)(\sigma_k \|g^k\|)^2 < \infty$.

(iii) For each limit point u^* of $\{u^k\}$ such that $u^* \in U^d$ we have

$$\lim_{k\to\infty} u^k = u^*.$$

Note that the sequence $\{u^k\}$ has limit points due to (i). Thus, due to (iii), it suffices to show that there exists a limit point of $\{u^k\}$ which belongs to U^d. However, process (11), (12) is only a conceptual scheme, since it does not contain the rules of choosing the parameters g^k and σ_k satisfying (12). This approach was first proposed by Konnov [1], where it was also noticed that iterations of most relaxation methods can be applied to find these parameters (see also [5, 8] for more details).

1.4 An Implementable CR Method

The blanket assumptions are the following.

- *U is a nonempty, closed and convex subset of R^n;*
- *V is a closed convex subset of R^n such that $U \subseteq V$;*
- *$G : V \to R^n$ is a locally Lipschitz continuous mapping;*
- *$U^* = U^d \neq \emptyset$.*

Method 1.1. *Step 0 (Initialization)*: Choose a point $u^0 \in U$, a family of mappings $\{T_k\}$ satisfying (A1) – (A3) with $V = U$ and a sequence of mappings $\{P_k\}$, where $P_k \in \mathcal{F}(U)$ for $k = 0, 1, \ldots$ Choose numbers $\alpha \in (0,1)$, $\beta \in (0,1)$, $\gamma \in (0,2)$, $\tilde{\theta} > 0$. Set $k := 0$.

Step 1 (Auxiliary procedure):

Step 1.1 : Solve the auxiliary variational inequality problem of finding $z^k \in U$ such that

$$\langle G(u^k) + T_k(u^k, z^k), v - z^k \rangle \geq 0 \quad \forall v \in U \tag{14}$$

and set $p^k := z^k - u^k$. If $p^k = 0$, stop.

Step 1.2: Determine m as the smallest number in Z_+ such that

$$u^k + \beta^m \tilde{\theta} p^k \in U, \langle G(u^k + \beta^m \tilde{\theta} p^k), p^k \rangle \leq \alpha \langle G(u^k), p^k \rangle, \tag{15}$$

set $\theta_k := \beta^m \tilde{\theta}$, $v^k := u^k + \theta_k p^k$. If $G(v^k) = 0$, stop.

Step 2 (Main iteration): Set

$$g^k := G(v^k), \sigma_k := \langle G(v^k), u^k - v^k \rangle / \|g^k\|^2, u^{k+1} := P_k(u^k - \gamma \sigma_k g^k), \tag{16}$$

$k := k + 1$ and go to Step 1.

According to the description, at each iteration we solve the auxiliary problem (7) with $\lambda_k = 1$ and carry out an Armijo-Goldstein type linesearch procedure. Thus, our method requires no a priori information about the original problem (1). In particular, it does not use the Lipschitz constant for G.

Theorem 1.1 *Let a sequence $\{u^k\}$ be generated by Method 1.1. Then:*
(i) If the method terminates at Step 1.1 (Step 1.2) of the kth iteration, $u^k \in U^*$ ($v^k \in U^*$).
(ii) If $\{u^k\}$ is infinite, we have

$$\lim_{k \to \infty} u^k = u^* \in U^*.$$

Theorem 1.2 *Let an infinite sequence $\{u^k\}$ be constructed by Method 1.1. Suppose that G is strongly monotone. Then:*
(i) the sequence $\{\|u^k - \pi_{U^*}(u^k)\|\}$ converges to zero in the rate $O(1/\sqrt{k})$;
(ii) if $U = R^n$, $\{\|u^k - \pi_{U^*}(u^k)\|\}$ converges to zero in a linear rate.

We now give conditions that ensure finite termination of the method. Namely, let us consider the following assumption.

(A4) *There exists a number $\mu' > 0$ such for each point $u \in U$, the following inequality holds:*

$$\langle G(u), u - \pi_{U^*}(u) \rangle \geq \mu' \, \|u - \pi_{U^*}(u)\|. \tag{17}$$

Theorem 1.3 *Let a sequence $\{u^k\}$ be constructed by Method 1.1. Suppose that (A4) holds. Then the method terminates with a solution.*

1.5 Modifications

In Step 1 of Method 1.1, we solved the auxiliary problem (14), which corresponds to (7) with $\lambda_k = 1$, and afterwards found the stepsize along the ray $u^k + \theta(z^k - u^k)$. However, it is clear that one can satisfy condition (15) by sequential solving problem

(7) with various λ_k. We now describe a CR method which involves a modified linesearch procedure.

Method 1.2. *Step 0 (Initialization)*: Choose a point $u^0 \in U$, a family of mappings $\{T_k\}$ satisfying (A1) – (A3) with $V = U$ and a sequence of mappings $\{P_k\}$, where $P_k \in \mathcal{F}(U)$ for $k = 0, 1, \ldots$ Choose numbers $\alpha \in (0,1)$, $\beta \in (0,1)$, $\gamma \in (0,2)$, $\tilde{\theta} > 0$. Set $k := 0$.

Step 1 (Auxiliary procedure):

Step 1.1 : Find m as the smallest number in Z_+ such that

$$\langle G(z^{k,m}), z^{k,m} - u^k \rangle \leq \alpha \langle G(u^k), z^{k,m} - u^k \rangle, \tag{18}$$

where $z^{k,m}$ is a solution of the auxiliary problem of finding $\bar{z} \in U$ such that

$$\langle G(u^k) + (\tilde{\theta}\beta^m)^{-1} T_k(u^k, \bar{z}), u - \bar{z} \rangle \geq 0 \quad \forall u \in U. \tag{19}$$

Step 1.2: Set $\theta_k := \beta^m \tilde{\theta}$, $v^k := z^{k,m}$. If $u^k = v^k$ or $G(v^k) = 0$, stop.

Step 2 (Main iteration): Set

$$g^k := G(v^k), \sigma_k := \langle G(v^k), u^k - v^k \rangle / \|g^k\|^2, \tag{20}$$

$$u^{k+1} := P_k(u^k - \gamma \sigma_k g^k), \tag{21}$$

$k := k + 1$ and go to Step 1.

Method 1.2 possesses the same convergence properties as those of Method 1.1.

We now describe a CR method which uses a different rule of computing the descent direction.

Method 1.3. *Step 0 (Initialization)*: Choose a point $u^0 \in V$, a family of mappings $\{T_k\}$ satisfying (A1) – (A3), and choose a sequence of mappings $\{P_k\}$, where $P_k \in \mathcal{F}(V)$, for $k = 0, 1, \ldots$ Choose numbers $\alpha \in (0,1)$, $\beta \in (0,1)$, $\gamma \in (0,2)$, $\tilde{\theta} > 0$. Set $k := 0$.

Step 1 (Auxiliary procedure):

Step 1.1 : Find m as the smallest number in Z_+ such that

$$\langle G(u^k) - G(z^{k,m}), u^k - z^{k,m} \rangle \leq (1 - \alpha)(\tilde{\theta}\beta^m)^{-1} \langle T_k(u^k, z^{k,m}), z^{k,m} - u^k \rangle, \tag{22}$$

where $z^{k,m}$ is a solution of the auxiliary problem of finding $\bar{z} \in U$ such that (19) holds.

Step 1.2: Set $\theta_k := \beta^m \tilde{\theta}$, $v^k := z^{k,m}$. If $u^k = v^k$ or $G(v^k) = 0$, stop.

Step 2 (Main iteration): Set

$$g^k := G(v^k) - G(u^k) - \theta_k^{-1} T_k(u^k, v^k), \tag{23}$$

$$\sigma_k := \langle g^k, u^k - v^k \rangle / \|g^k\|^2, \tag{24}$$

$$u^{k+1} := P_k(u^k - \gamma \sigma_k g^k), \tag{25}$$

$k := k + 1$ and go to Step 1.

Theorem 1.4 *Let a sequence* $\{u^k\}$ *be generated by Method 1.2. Then:*

(i) If the method terminates at the kth iteration, $v^k \in U^*$*.*

(ii) If $\{u^k\}$ *is infinite, we have*

$$\lim_{k\to\infty} u^k = u^* \in U^*.$$

Theorem 1.5 *Let an infinite sequence* $\{u^k\}$ *be generated by Method 1.3. If* G *is strongly monotone, then* $\{\|u^k - \pi_{U^*}(u^k)\|\}$ *converges to zero in a linear rate.*

Theorem 1.6 *Let a sequence* $\{u^k\}$ *be constructed by Method 1.3. Suppose that (A4) holds. Then the method terminates with a solution.*

1.6 CR Method for Nonlinearly Constrained Problems

In this section, we consider the case of VI (1) subject to nonlinear constraints. More precisely, we suppose that

$$U = \{u \in R^n \mid h_i(u) \le 0 \quad i = 1, \dots, m\},$$

where $h_i : R^n \to R$, $i = 1, \dots, m$ are Lipschitz continuously differentiable and convex functions, there exists a point $\bar{u}$ such that $h_i(\bar{u}) < 0$ for $i = 1, \dots, m$.

Since the functions h_i, $i = 1, \dots, m$ need not be affine, the auxiliary problems at Step 1 of the previous CR methods cannot in general be solved by finite algorithms. Therefore, we need a finite auxiliary procedure for this case. Such a CR method can be described as follows. Set

$$I_\varepsilon(u) = \{i \mid 1 \le i \le m, \quad h_i(u) \ge -\varepsilon\}.$$

Method 1.4. *Step 0 (Initialization)*: Choose a point $u^0 \in U$, sequences $\{\varepsilon_l\}$ and $\{\eta_l\}$ such that

$$\{\varepsilon_l\} \searrow 0, \{\eta_l\} \searrow 0. \tag{26}$$

Also, choose a sequence of mappings $\{P_k\}$, where $P_k \in \mathcal{F}(U)$ for $k = 0, 1, \dots$ Choose numbers $\alpha \in (0, 1)$, $\beta \in (0, 1)$, $\gamma \in (0, 2)$, $\theta \in (0, 1]$, and $\mu_i > 0$ for $i = 1, \dots, m$. Set $l := 1$, $k := 0$.

Step 1 (Auxiliary procedure):

Step 1.1 : Find the solution $(\tau_{k,l}, p^{k,l})$ of the problem

$$\min \quad \to \quad \tau \tag{27}$$

subject to

$$\begin{array}{rcll} \langle G(u^k), p\rangle & \le & \tau, & \\ \langle \nabla h_i(u^k), p\rangle & \le & \mu_i \tau & \quad i \in I_{\varepsilon_l}(u^k), \\ |p_s| & \le & 1 & s = 1, \dots, n. \end{array} \tag{28}$$

Step 1.2: If $\tau_{k,l} \geq -\eta_l$, set $v^k := u^k$, $u^{k+1} := u^k$, $g^k := 0$, $\sigma_k := 0$, $k := k+1$, $l := l+1$ and go to Step 1. (*null step*)

Step 1.3: Determine m as the smallest number in Z_+ such that

$$u^k + \beta^m \tilde{\theta} p^{k,l} \in U, \langle G(u^k + \beta^m \tilde{\theta} p^{k,l}), p^{k,l} \rangle \leq \alpha \langle G(u^k), p^{k,l} \rangle, \tag{29}$$

set $\theta_k := \beta^m \tilde{\theta}$, $v^k := u^k + \theta_k p^{k,l}$. (*descent step*)

Step 2 (Main iteration): Set

$$g^k := G(v^k), \sigma_k := \langle G(v^k), u^k - v^k \rangle / \|g^k\|^2, u^{k+1} := P_k(u^k - \gamma \sigma_k g^k), \tag{30}$$

$k := k+1$ and go to Step 1.

It is easy to see that the auxiliary procedure in Step 1 is an analogue of an iteration of the Zoutendijk feasible direction method.

Theorem 1.7 *Let a sequence $\{u^k\}$ be generated by Method 1.4. Then*

$$\lim_{k \to \infty} u^k = u^* \in U^*.$$

The procedure for implementing a f.n.e. mapping P_k is for example the following (see [12]). Set

$$I_+(u) = \{i \mid 1 \leq i \leq m, \quad h_i(u) > 0\}.$$

Procedure P. *Data:* A point $v \in R^n$.

Output: A point p.

Step 0: Set $w^0 := v$, $j := 0$.

Step 1: If $w^j \in U$, set $p := w^j$ and stop.

Step 2: Choose $i(j) \in I_+(w^j)$, $q^j \in \partial h_{i(j)}(w^j)$, set

$$w^{j+1} := w^j - 2h_{i(j)}(w^j) q^j / \|q^j\|^2, \tag{31}$$

$j := j+1$ and go to Step 1.

Procedure P is a variant of the reflection method generalizing the relaxation method for solving linear inequalities. Indeed, the points w^j and w^{j+1} in (31) are symmetric with respect to the hyperplane

$$H_j = \{u \in R^n \mid \langle q^j, u - w^j \rangle = -h_{i(j)}(w^j)\},$$

which separates w^j and U. For other approaches to construct f.n.e. mappings see [11].

2 Variational Inequalities with Multivalued Mappings

We will consider combined relaxation (CR) methods for solving variational inequalities which involve a multivalued mapping. This approach was suggested and developed in [2, 3, 6, 10].

2.1 Problem Formulation

Let U be a nonempty, closed and convex subset of the n-dimensional Euclidean space R^n, $G : U \to \Pi(R^n)$ a multivalued mapping. The *generalized variational inequality problem* (GVI for short) is the problem of finding a point $u^* \in U$ such that

$$\exists g^* \in G(u^*), \quad \langle g^*, u - u^* \rangle \geq 0 \quad \forall u \in U. \tag{1}$$

The solution of GVI (1) is closely related with that of the corresponding *dual generalized variational inequality problem* (DGVI for short), which is to find a point $u^* \in U$ such that

$$\forall\, u \in U \text{ and } \forall g \in G(u) : \langle g, u - u^* \rangle \geq 0. \tag{2}$$

We denote by U^* (respectively, by U^d) the solution set of problem (1) (respectively, problem (2)).

Definition 2.1 Let W and V be convex sets in R^n, $W \subseteq V$, and let $Q : V \to \Pi(R^n)$ be a multivalued mapping. The mapping Q is said to be

(a) a *K-mapping* on W, if it is u.s.c. on W and has nonempty convex and compact values;

(b) *u-hemicontinuous* on W, if for all $u \in W$, $v \in W$ and $\alpha \in [0, 1]$, the mapping $\alpha \mapsto \langle T(u + \alpha w), w \rangle$ with $w = v - u$ is u.s.c. at 0^+.

Now we give an extension of the Minty Lemma for the multivalued case.

Proposition 2.1

(i) The set U^d is convex and closed.

(ii) If G is u-hemicontinuous and has nonempty convex and compact values, then $U^d \subseteq U^*$.

(iii) If G is pseudomonotone, then $U^* \subseteq U^d$.

The existence of solutions to DGVI will play a crucial role for convergence of CR methods for GVI. Note that the existence of a solution to (2) implies that (1) is also solvable under mild assumptions, whereas the reverse assertion needs generalized monotonicity assumptions.

2.2 CR Method for the Generalized Variational Inequality Problem

We now consider a method for solving GVI (1). The blanket assumptions of this section are the following:

- *U is a subset of R^n, which is defined by*
$$U = \{u \in R^n \mid h(u) \le 0\}, \tag{3}$$
where $h : R^n \to R$ is a convex, but not necessarily differentiable, function;
- *the Slater condition is satisfied, i.e., there exists a point $\bar{u}$ such that $h(\bar{u}) < 0$;*
- *$G : U \to \Pi(R^n)$ is a K-mapping;*
- *$U^* = U^d \neq \emptyset$.*

Let us define the mapping $Q : R^n \to \Pi(R^n)$ by

$$Q(u) = \begin{cases} G(u) & \text{if } h(u) \le 0, \\ \partial h(u) & \text{if } h(u) > 0. \end{cases} \tag{4}$$

Method 2.1. *Step 0 (Initialization)*: Choose a point $u^0 \in U$, bounded positive sequences $\{\varepsilon_l\}$ and $\{\eta_l\}$. Also, choose numbers $\theta \in (0,1)$, $\gamma \in (0,2)$, and a sequence of mappings $\{P_k\}$, where $P_k \in \mathcal{F}(U)$ for $k = 0, 1, \dots$ Set $k := 0$, $l := 1$.

Step 1 (Auxiliary procedure) :

Step 1.1 : Choose q^0 from $Q(u^k)$, set $i := 0$, $p^i := q^i$, $w^0 := u^k$.

Step 1.2: If
$$\|p^i\| \le \eta_l, \tag{5}$$
set $y^l := u^{k+1} := u^k$, $k := k+1$, $l := l+1$ and go to Step 1. (*null step*)

Step 1.3: Set $w^{i+1} := u^k - \varepsilon_l p^i / \|p^i\|$, choose $q^{i+1} \in Q(w^{i+1})$. If
$$\langle q^{i+1}, p^i \rangle > \theta \|p^i\|^2, \tag{6}$$
then set $v^k := w^{i+1}$, $g^k := q^{i+1}$, and go to Step 2. (*descent step*)

Step 1.4: Set
$$p^{i+1} := \mathrm{Nr}\ \mathrm{conv}\{p^i, q^{i+1}\}, \tag{7}$$
$i := i+1$ and go to Step 1.2.

Step 2 (Main iteration): Set $\sigma_k := \langle g^k, u^k - v^k \rangle / \|g^k\|^2$,

$$u^{k+1} := P_k(u^k - \gamma \sigma_k g^k),$$

$k := k + 1$ and go to Step 1.

According to the description, at each iteration, the auxiliary procedure in Step 1 is applied for direction finding. In the case of a null step, the tolerances ε_l and η_l decrease since the point u^k approximates a solution within ε_l, η_l. Hence, the variable l is a counter for null steps and the variable $j(\cdot)$ is a counter for descent steps. In the case of a descent step we must have $\sigma_k > 0$. Therefore, the point $\tilde{u}^{k+1} = u^k - \gamma\sigma_k g^k$ is the projection of the point u^k onto the hyperplane

$$H_k(\gamma) = \{v \in R^n \mid \langle g^k, v - u^k \rangle = -\gamma\sigma_k \|g^k\|^2\}.$$

Clearly, $H_k(1)$ separates u^k and U^d. Hence, the distance from $\tilde{u}^{k+1}$ to each point of U^d cannot increase when $\gamma \in (0, 2)$ and that from u^{k+1} does so due to the properties of P_k. Thus, our method follows the general CR framework.

We will call one increase of the index i an inner step, so that the number of inner steps gives the number of computations of elements from $Q(\cdot)$ at the corresponding points.

Theorem 2.1 *Let a sequence $\{u^k\}$ be generated by Method 2.1 and let $\{\varepsilon_l\}$ and $\{\eta_l\}$ satisfy the following relations:*

$$\{\varepsilon_l\} \searrow 0, \{\eta_l\} \searrow 0. \tag{8}$$

Then:

(i) The number of inner steps at each iteration is finite.

(ii) It holds that

$$\lim_{k\to\infty} u^k = u^* \in U^*.$$

As Method 2.2 has a two-level structure, each iteration containing a finite number of inner steps, it is more suitable to derive its complexity estimate, which gives the total amount of work of the method. We use the distance to u^* as an accuracy function for our method, i.e., our approach is slightly different from the standard ones. More precisely, given a starting point u^0 and a number $\delta > 0$, we define the complexity of the method, denoted by $N(\delta)$, as the total number of inner steps t which ensures finding a point $\bar{u} \in U$ such that

$$\|\bar{u} - u^*\| / \|u^0 - u^*\| \leq \delta.$$

Therefore, since the computational expense per inner step can easily be evaluated for each specific problem under examination, this estimate in fact gives the total amount of work. We thus proceed to obtain an upper bound for $N(\delta)$.

Theorem 2.2 *Suppose G is monotone and there exists $u^* \in U^*$ such that*

$$\begin{array}{c} \text{for every } u \in U \text{ and for every } g \in G(u), \\ \langle g, u - u^* \rangle \geq \mu \|u - u^*\|, \end{array} \qquad (9)$$

for some $\mu > 0$. Let a sequence $\{u^k\}$ be generated by Method 2.1 where

$$\varepsilon_l = \nu^l \varepsilon', \eta_l = \eta', l = 0, 1, \ldots; \quad \nu \in (0, 1). \qquad (10)$$

Then, there exist some constants $\bar{\varepsilon} > 0$ and $\bar{\eta} > 0$ such that

$$N(\delta) \leq B_1 \nu^{-2} (\ln(B_0/\delta)/\ln \nu^{-1} + 1), \qquad (11)$$

where $0 < B_0, B_1 < \infty$, whenever $0 < \varepsilon' \leq \bar{\varepsilon}$ and $0 < \eta' \leq \bar{\eta}$, B_0 and B_1 being independent of ν.

It should be noted that the assertion of Theorem 2.2 remains valid without the additional monotonicity assumption on G if $U = R^n$.

Thus, our method attains a logarithmic complexity estimate, which corresponds to a linear rate of convergence with respect to inner steps. We now establish a similar upper bound for $N(\delta)$ in the single-valued case.

Theorem 2.3 *Suppose that $U = R^n$ and that G is strongly monotone and Lipschitz continuous. Let a sequence $\{u^k\}$ be generated by Method 2.2 where*

$$\varepsilon_l = \nu^l \varepsilon', \eta_l = \nu^l \eta', l = 0, 1, \ldots; \varepsilon' > 0, \eta' > 0; \quad \nu \in (0, 1). \qquad (12)$$

Then,

$$N(\delta) \leq B_1 \nu^{-6} (\ln(B_0/\delta)/\ln \nu^{-1} + 1), \qquad (13)$$

where $0 < B_0, B_1 < \infty$, B_0 and B_1 being independent of ν.

2.3 CR Method for Multivalued Inclusions

To solve GVI (1) , we propose to apply Method 2.1 to finding stationary points either of the mapping P being defined as follows

$$P(u) = \begin{cases} G(u) & \text{if } h(u) < 0, \\ \operatorname{conv}\{G(u) \bigcup \partial h(u)\} & \text{if } h(u) = 0, \\ \partial h(u) & \text{if } h(u) > 0. \end{cases} \qquad (14)$$

Such a method need not include feasible non-expansive operators and is based on the following observations.

First we note P in (14) is a K-mapping. Next, GVI (1) is equivalent to the multi-valued inclusion

$$0 \in P(u^*). \qquad (15)$$

We denote by S^* the solution set of problem (15).

Theorem 2.4 *It holds that*

$$U^* = S^*.$$

In order to apply Method 2.2 to problem (15) we have to show that its dual problem is solvable. Namely, let us consider the problem of finding a point u^* of R^n such that

$$\forall u \in R^n, \quad \forall t \in P(u), \quad \langle t, u - u^* \rangle \geq 0, \tag{16}$$

which can be viewed as the dual problem to (15). We denote by $S^*_{(d)}$ the solution set of this problem.

Lemma 2.1
(i) $S^*_{(d)}$ *is convex and closed.*
(ii) $S^*_{(d)} \subseteq S^*$.
(iii) If P *is pseudomonotone, then* $S^*_{(d)} = S^*$.

Note that P need not be pseudomonotone in general. Nevertheless, in addition to Theorem 2.4, it is useful to obtain the equivalence result for problems (2) and (16).

Proposition 2.2 $U^d = S^*_{(d)}$.

Therefore, we can apply Method 2.1 with replacing G, U, and P_k by P, R^n, and I respectively, to the multivalued inclusion (15) under the blanket assumptions. We call this modification Method 2.2.

Theorem 2.5 *Let a sequence* $\{u^k\}$ *be generated by Method 2.2 and let* $\{\varepsilon_l\}$ *and* $\{\eta_l\}$ *satisfy (8). Then:*
(i) The number of inner steps at each iteration is finite.
(ii) It holds that

$$\lim_{k\to\infty} u^k = u^* \in S^* = U^*.$$

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