THEORY AND APPLICATIONS OF VARIATIONAL INEQUALITIES

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0.1 Abstract

In this paper, we outline some results in theory of variational inequalities with the emphasis on monotonicity properties of cost mappings. Their relationships with other problems of Nonlinear Analysis and some applications are also discussed.

Key Words: Variational inequalities, monotonicity properties, applications.

1 Variational Inequalities with Continuous Mappings

We first consider the main results in theory of variational inequality problems with continuous single-valued mappings and their relationships with other general problems of Nonlinear Analysis.

1.1 Problem Formulation

Let U be a nonempty, closed and convex subset of the n-dimensional Euclidean space R^n, $G : U \to R^n$ a continuous mapping. The *variational inequality problem* (VI) is the problem of finding a point $u^* \in U$ such that

$$\langle G(u^*), u - u^* \rangle \geq 0 \quad \forall u \in U. \tag{1.1}$$

It is well known that the solution of VI (1.1) is closely related with that of the following problem of finding $u^* \in U$ such that

$$\langle G(u), u - u^* \rangle \geq 0 \quad \forall u \in U. \tag{1.2}$$

Problem (1.2) may be termed as the dual formulation of VI (DVI). We will denote by U^* (respectively, by U^d) the solution set of problem (1.1) (respectively, problem (1.2)).

To obtain relationships between U^* and U^d, we need additional monotonicity type properties of G.

Definition 1.1 Let W and V be convex sets in R^n, $W \subseteq V$, and let $Q : V \to R^n$ be a mapping. The mapping Q is said to be

(a) *strongly monotone* on W if

$$\langle Q(u) - Q(v), u - v \rangle \geq \tau \|u - v\|^2 \qquad \forall u, v \in W;$$

(b) *strictly monotone* on W if

$$\langle Q(u) - Q(v), u - v \rangle > 0 \qquad \forall u, v \in W, u \neq v;$$

(c) *monotone* on W if

$$\langle Q(u) - Q(v), u - v \rangle \geq 0 \qquad \forall u, v \in W;$$

(d) *pseudomonotone* on W if

$$\langle Q(v), u - v \rangle \geq 0 \quad \Longrightarrow \quad \langle Q(u), u - v \rangle \geq 0 \qquad \forall u, v \in W;$$

(e) *quasimonotone* on W if f

$$\langle Q(v), u - v \rangle > 0 \quad \Longrightarrow \quad \langle Q(u), u - v \rangle \geq 0 \qquad \forall u, v \in W;$$

(f) *explicitly quasimonotone* on W if it is quasimonotone on W and

$$\langle Q(v), u - v \rangle > 0 \quad \Longrightarrow \quad \langle Q(z), u - v \rangle > 0 \quad \exists z \in (0.5(u + v), u).$$

It follows from the definitions that the following implications hold:

$$(a) \Longrightarrow (b) \Longrightarrow (c) \Longrightarrow (d) \Longrightarrow (f) \Longrightarrow (e).$$

The reverse assertions are not true in general.

Now we recall the relationships between solution sets of VI and DVI, which are known as the Minty Lemma.

Proposition 1.1
(i) U^d is convex and closed.
(ii) $U^d \subseteq U^*$.
(iii) If G is pseudomonotone, $U^* \subseteq U^d$.

The existence of solutions of DVI plays a crucial role in constructing relatively simple solution methods for VI. We note that Proposition 1.1 (iii) does not hold in the (explicitly) quasimonotone case. Moreover, problem (1.2) may even have no solutions in the quasimonotone case. However, we can give an example of solvable DVI (1.2) with the underlying mapping G which is not quasimonotone. Nevertheless, as it was shown by Konnov (1998), explicitly quasimonotone DVI is solvable under the usual assumptions.

1.2 Variational Inequalities and Related Problems

VI's are closely related with many general problems of Nonlinear Analysis, such as complementarity, fixed point and optimization problems. The simplest example of VI is the problem of solving a system of equations. It is easy to see that if $U = R^n$ in (1.1), then VI (1.1) is equivalent to the problem of finding a point $u^* \in R^n$ such that

$$G(u^*) = 0.$$

If the mapping G is affine, i.e., $G(u) = Mu + q$, then the above problem is equivalent to the classical system of linear equations

$$Mu^* = -q. \tag{1.3}$$

Let U be a convex cone in R^n. The *complementarity problem* (CP for short) is to find a point $u^* \in U$ such that

$$G(u^*) \in U', \quad \langle G(u^*), u^* \rangle = 0, \tag{1.4}$$

where U' is the dual cone to U, i.e.,

$$U' = \{v \in R^n \mid \langle u, v \rangle \geq 0 \quad \forall u \in U\}.$$

This problem can be viewed as a particular case of VI, as stated below.

Proposition 1.2 *Let U be a convex cone. Then problem (1.1) is equivalent to problem (1.4).*

Among various classes of CP's, the following ones are most investigated. The *standard complementarity problem* corresponds to the case where $U = R^n_+$ in (1.4) and the *linear complementarity problem* (LCP for short) corresponds to the case where $U = R^n_+$ and G is affine, i.e., $G(u) = Mu + q$ in (1.4).

Next, let U be again an arbitrary convex closed set in R^n and let T be a continuous mapping from U into itself. The *fixed point problem* is to find a point $u^* \in U$ such that

$$u^* = T(u^*). \tag{1.5}$$

This problem can be also converted into a VI format.

Proposition 1.3 *If the mapping G is defined by*

$$G(u) = u - T(u), \tag{1.6}$$

then problem (1.1) coincides with problem (1.5).

Moreover, we can obtain the condition which guarantees the monotonicity of G.

Definition 1.2 A mapping $Q : R^n \to R^n$ is said to be *non-expansive* on V if for each pair of points $u, v \in V$, we have

$$\|Q(u) - Q(v)\| \leq \|u - v\|.$$

Proposition 1.4 *If $T : U \to U$ is non-expansive, then the mapping G, defined in (1.6), is monotone.*

Now, we consider the well-known optimization problem. Let $f : U \to R$ be a real-valued function. Then we can define the following *optimization problem* of finding a point $u^* \in U$ such that

$$f(u^*) \leq f(u) \quad \forall u \in U,$$

or briefly,

$$\min \to \{f(u) \mid u \in U\}. \tag{1.7}$$

We denote by U_f the solution set of this problem.

Recall the definitions of convexity type properties for functions.

Definition 1.3 Let W and V be convex sets in R^n such that $W \subseteq V$, and let $\varphi : V \to R$ be a differentiable function. The function φ is said to be

(a) *strongly convex* on W with constant $\tau > 0$ if for each pair of points $u, v \in W$ and for all $\alpha \in [0, 1]$, we have

$$\varphi(\alpha u + (1 - \alpha)v) \leq \alpha\varphi(u) + (1 - \alpha)\varphi(v) - 0.5\alpha(1 - \alpha)\tau\|u - v\|^2;$$

(b) *strictly convex* on W if for all distinct $u, v \in W$ and for all $\alpha \in (0, 1)$,

$$\varphi(\alpha u + (1 - \alpha)v) < \alpha\varphi(u) + (1 - \alpha)\varphi(v);$$

(c) *convex* on W if for each pair of points $u, v \in W$ and for all $\alpha \in [0, 1]$, we have

$$\varphi(\alpha u + (1 - \alpha)v) \leq \alpha\varphi(u) + (1 - \alpha)\varphi(v);$$

(d) *pseudoconvex* on W if for each pair of points $u, v \in W$ and for all $\alpha \in [0, 1]$, we have

$$\langle \nabla\varphi(v), u - v\rangle \geq 0 \quad \text{implies } \varphi(u) \geq \varphi(v);$$

(e) *quasiconvex* on W if for each pair of points $u, v \in W$ and for all $\alpha \in [0, 1]$, we have

$$\varphi(\alpha u + (1 - \alpha)v) \leq \max\{\varphi(u), \varphi(v)\};$$

(f) *explicitly quasiconvex* on W if it is quasiconvex on W and for all distinct $u, v \in W$ and for all $\alpha \in (0,1)$, we have

$$\varphi(\alpha u + (1-\alpha)v) < \max\{\varphi(u), \varphi(v)\}.$$

The function $\varphi : V \to R$ is said to be *strongly concave* with constant τ (respectively, *strictly concave, pseudoconcave, quasiconcave, explicitly quasiconcave*) on W, if the function $-\varphi$ is strongly convex with constant τ (respectively, strictly convex, convex, pseudoconvex, quasiconvex, explicitly quasiconvex) on W.

It follows directly from the definitions that the following implications hold:

$$(a) \Longrightarrow (b) \Longrightarrow (c) \Longrightarrow (f) \Longrightarrow (e).$$

The reverse assertions are not true in general. We can also include the pseudoconvexity in the above relations. The implications $(c) \Longrightarrow (d)$ and $(d) \Longrightarrow (f)$ are also true. We now state the relationships between (generalized) convexity of functions and (generalized) monotonicity of their gradients.

Proposition 1.5 *Let W be an open convex subset of V. A differentiable function $f : V \to R$ is strongly convex with constant τ (respectively, strictly convex, convex, pseudoconvex, quasiconvex, explicitly quasiconvex) on W, if and only if its gradient map $\nabla f : U \to R^n$ is strongly monotone with constant τ (respectively, strictly monotone, monotone, pseudomonotone, quasimonotone, explicitly monotone) on W.*

We now give the well-known optimality condition for problem (1.7).

Theorem 1.1 *Suppose that $f : U \to R$ is a differentiable function. Then:*
(i) $U_f \subseteq U^*$, i.e., each solution of (1.7) is a solution of VI (1.1), where

$$G(u) = \nabla f(u); \tag{1.8}$$

(ii) if f is pseudoconvex and G is defined by (1.8), then $U^* \subseteq U_f$.

Thus, optimization problem (1.7) can be reduced to VI (1.1) with the (generalized) monotone underlying mapping G if the function f in (1.7) possesses the corresponding (generalized) convexity property. However, VI which expresses the optimality condition in optimization enjoys additional properties in comparison with the usual VI. For instance, if f is twice continuously differentiable, then its Hessian matrix $\nabla^2 f = \nabla G$ is symmetric. Conversely, if the mapping $\nabla G : R^n \to R^n \times R^n$ is symmetric, then for any fixed v there exists the function

$$f(u) = \int_0^1 \langle G(v + \tau(u-v)), u - v \rangle d\tau$$

such that (1.8) holds. It is obvious that the Jacobian ∇G in (1.1) is in general asymmetric. Next, consider the case of convex optimization problem (1.7). In other words, let the function f be convex and differentiable. Then, according to Theorem 1.1, (1.7) is equivalent to (1.1) with G being defined in (1.8). Due to Proposition 1.5, the mapping ∇f is monotone. Besides, for each $u \in U \backslash U^*$, we have

$$\langle \nabla f(u), u - u^* \rangle > 0; \tag{1.9}$$

i.e., $-\nabla f(u)$ makes an acute angle with any vector $u^* - u$ at each non optimal point $u \in U$. In the general case this property does not hold. Namely, consider problem (1.1) with G being monotone. Then, at each $u \in U \backslash U^*$, we only have

$$\langle G(u), u - u^* \rangle \geq 0$$

due to Proposition 1.1, i.e., the angle between $-G(u)$ and $u - u^*$ need not be acute. We now give the classical example of such a problem.

Example 1.1 Let $U = R^2$, $G(u) = (u_2, -u_1)^T$. Then G is monotone, $U^* = \{(0,0)^T\}$, but for any $u \notin U^*$ we have

$$\langle G(u), u - u^* \rangle = u_2 u_1 - u_1 u_2 = 0,$$

i.e., the property similar to (1.9) does not hold. It should be also noted that the Jacobian

$$\nabla G(u) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

is asymmetric, so that there is no function f satisfying (1.8).

Saddle point problems are closely related to optimization as well as to noncooperative game problems. Let X be a convex closed set in R^l and let Y be a convex closed set in R^m. Suppose that $L : R^l \times R^m \to R$ is a differentiable convex-concave function, i.e., $L(\cdot, y)$ is convex for each $y \in Y$ and $L(x, \cdot)$ is concave for each $x \in X$. The *saddle point problem* is to find a pair of points $x^* \in X$, $y^* \in Y$ such that

$$L(x^*, y) \leq L(x^*, y^*) \leq L(x, y^*) \quad \forall x \in X, \forall y \in Y. \tag{1.10}$$

Set $n = l + m$, $U = X \times Y$ and define the mapping $G : R^n \to R^n$ as follows:

$$G(u) = G(x, y) = \begin{pmatrix} \nabla_x L(x, y) \\ -\nabla_y L(x, y) \end{pmatrix}. \tag{1.11}$$

From Theorem 1.1 we now obtain the following equivalence result.

Corollary 1.1 *Problems (1.10) and (1.1), (1.11) are equivalent.*

It should be noted that G in (1.11) is also monotone.

Saddle point problems are proved to be a useful tool for "eliminating" functional constraints in optimization. Let us consider the optimization problem

$$\min \to \{f_0(x) \mid x \in D\}, \tag{1.12}$$

where

$$D = \{x \in X \mid f_i(x) \leq 0 \quad i = 1, \dots, m\}, \tag{1.13}$$

$f_i : R^l \to R$, $i = 0, \dots, m$ are convex differentiable functions,

$$X = \{x \in R^l \mid x_j \geq 0 \quad \forall j \in J\}, J \subseteq \{1, \dots, l\}. \tag{1.14}$$

Then we can define the Lagrange function associate to problem (1.12) – (1.14) as follows:

$$L(x, y) = f_0(x) + \sum_{i=1}^{m} y_i f_i(x). \tag{1.15}$$

To obtain the relationships between problems (1.12) – (1.14) and (1.10), (1.15), we need certain constraint qualification conditions. Namely, consider the following assumption.

(C) *Either all the functions f_i, $i = 1, \dots, m$ are affine, or there exists a point $\bar{x}$ such that $f_i(\bar{x}) < 0$ for all $i = 1, \dots, m$.*

Proposition 1.6 *(i) If (x^*, y^*) is a saddle point of the function L in (1.15) with $Y = R^m_+$, then x^* is a solution to problem (1.12) – (1.14).*

(ii) If x^* is a solution to problem (1.12) – (1.14) and condition (C) holds, then there exists a point $y^* \in Y = R^m_+$ such that (x^*, y^*) is a solution to the saddle point problem (1.10), (1.15).

By using Corollary 1.2 we now see that optimization problem (1.12) – (1.14) can be replaced by VI (1.1) (or equivalently, by CP (1.4) since X is a convex cone), where $U = X \times Y$, $Y = R^m_+$, $f(x) = (f_1(x), \dots, f_m(x))^T$, and

$$G(u) = \begin{pmatrix} \nabla f_0(x) + \sum_{i=1}^{m} y_i \nabla f_i(x) \\ -f(x) \end{pmatrix} \tag{1.16}$$

with G being monotone.

2 Variational Inequalities with Multivalued Mappings

We now consider the main results in theory of variational inequality problems with continuous multivalued mappings and their relationships with other general problems of Nonlinear Analysis.

2.1 Problem Formulation

Let U be a nonempty, closed and convex subset of the n-dimensional Euclidean space R^n, $G : U \to \Pi(R^n)$ a multivalued mapping. The *generalized variational inequality problem* (GVI for short) is the problem of finding a point $u^* \in U$ such that

$$\exists g^* \in G(u^*), \quad \langle g^*, u - u^* \rangle \geq 0 \quad \forall u \in U. \tag{2.1}$$

The solution of GVI (2.1) is closely related with that of the corresponding *dual generalized variational inequality problem* (DGVI for short), which is to find a point $u^* \in U$ such that

$$\forall\, u \in U \text{ and } \forall g \in G(u) : \langle g, u - u^* \rangle \geq 0. \tag{2.2}$$

We denote by U^* (respectively, by U^d) the solution set of problem (2.1) (respectively, problem (2.2)).

Definition 2.1 Let W and V be convex sets in R^n, $W \subseteq V$, and let $Q : V \to \Pi(R^n)$ be a multivalued mapping. The mapping Q is said to be

(a) a *K-mapping* on W, if it is u.s.c. on W and has nonempty convex and compact values;

(b) *u-hemicontinuous* on W, if for all $u \in W$, $v \in W$ and $\alpha \in [0, 1]$, the mapping $\alpha \mapsto \langle T(u + \alpha w), w \rangle$ with $w = v - u$ is u.s.c. at 0^+.

Recall some definitions of monotonicity type properties for multivalued mappings.

Definition 2.2 Let W and V be convex sets in R^n, $W \subseteq V$, and let $Q : V \to \Pi(R^n)$ be a multivalued mapping. The mapping Q is said to be

(a) *strongly monotone* on W with constant $\tau > 0$ if for each pair of points $u, v \in W$ and for all $q' \in Q(u)$, $q'' \in Q(v)$, we have

$$\langle q' - q'', u - v \rangle \geq \tau \|u - v\|^2;$$

(b) *strictly monotone* on W if for all distinct $u, v \in W$ and for all $q' \in Q(u)$, $q'' \in Q(v)$, we have

$$\langle q' - q'', u - v \rangle > 0;$$

(c) *monotone* on W if for each pair of points $u, v \in W$ and for all $q' \in Q(u)$, $q'' \in Q(v)$, we have

$$\langle q' - q'', u - v \rangle \geq 0;$$

(d) *pseudomonotone* on W if for each pair of points $u, v \in W$ and for all $q' \in Q(u)$, $q'' \in Q(v)$, we have

$$\langle q'', u - v \rangle \geq 0 \quad \text{implies } \langle q', u - v \rangle \geq 0;$$

(e) *quasimonotone* on W if for each pair of points $u, v \in W$ and for all $q' \in Q(u)$, $q'' \in Q(v)$, we have

$$\langle q'', u - v \rangle > 0 \quad \text{implies } \langle q', u - v \rangle \geq 0;$$

(f) *explicitly quasimonotone* on W if it is quasimonotone on W and for all distinct $u, v \in W$ and for all $q' \in Q(u)$, $q'' \in Q(v)$, the relation

$$\langle q'', u - v \rangle > 0$$

implies

$$\langle q, u - v \rangle > 0 \text{ for some } q \in Q(z), z \in (0.5(u + v), u).$$

From the definitions we obviously have the following implications:

$$(a) \Longrightarrow (b) \Longrightarrow (c) \Longrightarrow (d) \Longrightarrow (e) \text{ and } (f) \Longrightarrow (e).$$

The reverse assertions are not true in general.

Now we give an extension of the Minty Lemma for the multivalued case.

Proposition 2.1 *(i) The set U^d is convex and closed.*

(ii) If G is u-hemicontinuous and has nonempty convex and compact values, then $U^d \subseteq U^*$.

(iii) If G is pseudomonotone, then $U^* \subseteq U^d$.

The existence of solutions to DGVI will play a crucial role for convergence of CR methods for GVI. Note that the existence of a solution to (2.2) implies that (2.1) is also solvable under mild assumptions, whereas the reverse assertion needs generalized monotonicity assumptions.

2.2 Generalized Variational Inequalities and Related Problems

The relationships between GVI's and other general problems of Nonlinear Analysis, which involve multivalued mappings, are in general the same as in the single-valued case. Namely, in the case of $U = R^n$, GVI (2.1) reduces to the following *multivalued inclusion problem*: Find $u^* \in R^n$ such that

$$0 \in G(u^*).$$

Next, suppose that U is a convex cone in R^n and consider the following *generalized complementarity problem* (GCP for short): Find $u^* \in U$ such that

$$\exists g^* \in G(u^*), \quad g^* \in U', \quad \langle g^*, u^* \rangle = 0. \tag{2.3}$$

This problem is in fact a particular case of GVI (2.1).

Proposition 2.2 *Let U be a convex cone. Then problem (2.1) is equivalent to problem (2.3).*

Let U be again an arbitrary convex closed set in R^n and let T be a multivalued mapping from U into itself. The *multivalued fixed point problem* associated to the mapping T can be defined as follows: Find $u^* \in U$ such that

$$u^* \in T(u^*). \tag{2.4}$$

The fixed point problem (2.4) can be also converted into a GVI format as stated below in the multivalued version of Proposition 1.3.

Proposition 2.3 *If the mapping G is defined by $G(u) = u - T(u)$, then problem (2.1) coincides with problem (2.4).*

The relationships between the optimization problem (1.7) and GVI (2.1) is the same as those in the differentiable case, with the substitution of the gradient of f with its subdifferential.

Definition 2.3 A bifunction $\Phi : U \times U \to R \bigcup \{+\infty\}$ is called an *equilibrium bifunction*, if $\Phi(u, u) = 0$ for each $u \in U$.

Let U be a nonempty, closed and convex subset of R^n and let $\Phi : U \times U \to R \bigcup \{+\infty\}$ be an equilibrium bifunction. The *equilibrium problem* (EP for short) is the problem of finding a point $u^* \in U$ such that

$$\Phi(u^*, v) \geq 0 \quad \forall v \in U. \tag{2.5}$$

EP gives a suitable and general format for various problems in Game Theory, Mathematical Physics, Economics and other fields. It is easy to see that EP (2.5) can be viewed as some extension of GVI (2.1). In fact, set

$$\Phi(u, v) = \sup_{g \in G(u)} \langle g, v - u \rangle. \tag{2.6}$$

Then each solution of (2.1) is a solution of (2.5), (2.6). The reverse assertion is true if G has convex and compact values. Further, by analogy with the dual variational inequality (2.2), we can define the following *dual equilibrium problem*: Find $v^* \in U$ such that

$$\Phi(u, v^*) \leq 0 \quad \forall u \in U. \tag{2.7}$$

We denote by U^0 and $U^0_{(d)}$ the solution sets of problems (2.5) and (2.7), respectively. The following existence result for EP was obtained by Ky Fan and extended by H. Brézis, L. Nirenberg and G. Stampacchia.

Proposition 2.4 *Let $\Phi : U \times U \to R\bigcup\{+\infty\}$ be an equilibrium bifunction such that $\Phi(\cdot, v)$ is u.s.c. for each $v \in U$ and $\Phi(u, \cdot)$ is quasiconvex for each $u \in U$. Suppose also that at least one of the following assumptions hold:*

(a) U is bounded;

(b) there exists a nonempty bounded subset W of U such that for every $u \in U \setminus W$ there is $v \in W$ with $\Phi(u, v) < 0$.

Then EP (2.5) has a solution.

In order to obtain relationships between U^0 and $U^0_{(d)}$, one usually needs monotonicity type conditions on Φ, which can be viewed as some extensions of those in Definition 2.2.

Definition 2.4 Let W and V be convex sets in R^n, $W \subseteq V$, and let $\Phi : V \times V \to R\bigcup\{+\infty\}$ be an equilibrium bifunction. The bifunction Φ is said to be

(a) *strongly monotone* on W with constant $\tau > 0$ if for each pair of points $u, v \in W$, we have

$$\Phi(u, v) + \Phi(v, u) \leq -\tau\|u - v\|^2;$$

(b) *strictly monotone* on W if for all distinct $u, v \in W$, we have

$$\Phi(u, v) + \Phi(v, u) < 0;$$

(c) *monotone* on W if for each pair of points $u, v \in W$, we have

$$\Phi(u, v) + \Phi(v, u) \leq 0;$$

(d) *pseudomonotone* on W if for each pair of points $u, v \in W$, we have

$$\Phi(u, v) \geq 0 \quad \text{implies } \Phi(v, u) \leq 0;$$

(e) *quasimonotone* on W if for each pair of points $u, v \in W$, we have

$$\Phi(u, v) > 0 \quad \text{implies } \Phi(v, u) \leq 0;$$

(f) *explicitly quasimonotone* on W if it is quasimonotone on W and for all distinct $u, v \in W$, the relation

$$\Phi(u, v) > 0$$

implies

$$\Phi(z, u) < 0 \text{ for some } z \in (0.5(u + v), v).$$

It is easy to see that properties (a) – (f) coincide with the corresponding properties (a) – (f) in Definition 2.2 provided (2.6) holds and G has convex and compact values. Besides, from the definitions above we obviously have the following implications:

$$(a) \Longrightarrow (b) \Longrightarrow (c) \Longrightarrow (d) \Longrightarrow (e) \text{ and } (f) \Longrightarrow (e).$$

The reverse assertions are not true in general.

The following assertion is an extension of the Minty Lemma for EP's.

Proposition 2.5 *Let $\Phi : U \times U \to R\bigcup\{+\infty\}$ be an equilibrium bifunction.*

(i) If $\Phi(u, \cdot)$ is quasiconvex and l.s.c. for each $u \in U$, then $U^0_{(d)}$ is convex and closed.

(ii) If $\Phi(\cdot, v)$ is u.s.c. for each $v \in U$, $\Phi(u, \cdot)$ is explicitly quasiconvex for each $u \in U$, then $U^0_{(d)} \subseteq U^0$.

(iii) If Φ is pseudomonotone, then $U^0 \subseteq U^0_{(d)}$.

Also, monotonicity properties enable one to obtain uniqueness results of solutions for EP.

Proposition 2.6 *Let $\Phi : U \times U \to R\bigcup\{+\infty\}$ be an equilibrium bifunction.*

(i) If Φ is strictly monotone, then EP (2.5) has at most one solution.

(ii) If $\Phi(\cdot, v)$ is u.s.c. for each $v \in U$, $\Phi(u, \cdot)$ is convex and l.s.c. for each $u \in U$, and Φ is strongly monotone, then EP (2.5) has a unique solution.

It is clear that the saddle point problem (1.10) is a particular case of EP. In fact, it suffices to set

$$\Phi(u, v) = L(x', y) - L(x, y'), u = (x, y)^T, v = (x', y')^T \tag{2.8}$$

and $U = X \times Y$ in (2.5). Note that Φ is obviously monotone and concave-convex in that case. In addition, if $L : X \times Y \to R$ is (strictly) strongly convex in x and (strictly) strongly concave in y, then Φ in (2.8) will be (strictly) strongly concave-convex. Next, the problem of solving a zero-sum two-person game is also a particular case of EP (2.5). Moreover, we can consider the general case of an m-person noncooperative game. Recall that such a game consists of m players, each of which has a strategy set $X_i \subseteq R^{n_i}$ and a utility function $f_i : U \to R$, where

$$U = X_1 \times \ldots \times X_m.$$

A point $u^* = (x^*_1, \ldots, x^*_m)^T \in U$ is said to be a *Nash equilibrium point* for this game, if

$$f_i(x^*_1, \ldots, x^*_{i-1}, y_i, x^*_{i+1}, \ldots, x^*_m)^T) \leq f_i(u^*) \quad \forall y_i \in X_i, i = 1, \ldots, m.$$

Set

$$\Psi(u, v) = -\sum_{i=1}^{m} f_i(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_m), \tag{2.9}$$

$$u = (x_1, \ldots, x_m)^T, v = (y_1, \ldots, y_m)^T,$$

and

$$\Phi(u, v) = \Psi(u, v) - \Psi(u, u), \tag{2.10}$$

then EP (2.5) becomes equivalent to the problem of finding Nash equilibrium points.

We can make use of GVI to formulate optimality conditions for EP. Let us consider EP (2.5) and set

$$G(u) = \partial\Phi_v(u, v)|_{v=u}. \tag{2.11}$$

We now present relationships between monotonicity type properties of Φ and G in (2.11).

Proposition 2.7 *Suppose that $\Phi : V \times V \to R\bigcup\{+\infty\}$ is an equilibrium bifunction such that $\Phi(u, \cdot)$ is convex and subdifferentiable for each $u \in U$. If Φ is strongly monotone with constant τ (respectively, strictly monotone, monotone, pseudomonotone, quasimonotone, explicitly quasimonotone) on U, then so is G in (2.11).*

Note that the reverse assertions are not true in general. Applying this result to the saddle point problem (1.10) and using (2.8), we now see that the mapping G in (2.11) will be monotone if L is convex-concave. This is also the case if we consider the convex optimization problem (1.7), (1.13), (1.14).

Theorem 2.1 *Let W be an open convex set in R^n such that $U \subseteq W$. Let $\Phi : W \times W \to R\bigcup\{+\infty\}$ be an equilibrium bifunction and let the mapping G be defined by (2.11).*

(i) If $\Phi(u, \cdot)$ is quasiconvex and regular for each $u \in W$, then

$$U^0 \subseteq U^*, U^0_{(d)} \subseteq U^d.$$

(ii) If $\Phi(u, \cdot)$ is pseudoconvex and l.s.c. for each $u \in W$, then

$$U^0 = U^*, U^0_{(d)} \subseteq U^d.$$

3 Applications

In this section, we consider some applications of VIs to economic and transportation equilibrium problems.

3.1 Economic Equilibrium Problems

A number of economic models are adjusted to investigating the conditions which balance supply and demand of commodities, i.e., they are essentially equilibrium models. As a rule, the concept of economic equilibrium can be written in terms of a complementarity relation between the price and the excess demand for each commodity. Therefore, most existing economic equilibrium models can be written as complementarity (or variational inequality) problems. To illustrate this assertion, we now describe one of the most general economic models originated by L. Walras.

So, it is assumed that our economy deals in n commodities and that there are m economics agents dealing with these commodities. Let $M = \{1, \dots, m\}$. We divide M into two subsets M_s and M_c which correspond to sectors (producers) and consumers, respectively. Given a price vector $p \in R^n_+$, the jth sector determines its supply set $S_j(p) \subseteq R^n_+$ and the ith consumer determines its demand set $D_i(p) \subseteq R^n_+$. Set

$$S(p) = \sum_{j \in M_s} S_j(p), D(p) = \sum_{i \in M_c} D_i(p).$$

Then we can define the excess demand mapping

$$E(p) = D(p) - S(p).$$

A vector p^* is said to be an *equilibrium price* if it satisfies the following conditions:

$$p^* \in R^n_+; \quad \exists q^* \in E(p^*): \quad -q^* \in R^n_+, \quad \langle q^*, p^* \rangle = 0. \tag{3.1}$$

We denote by P^* the whole set of equilibrium prices. If we now set

$$G(p) = -E(p),$$

then problem (3.1) obviously coincides with GCP (2.3) or equivalently, with the following GVI: find $p^* \in R^n_+$ such that

$$\exists g^* \in G(p^*), \quad \langle g^*, p - p^* \rangle \geq 0 \quad \forall p \in R^n_+; \tag{3.2}$$

see Proposition 2.2. Therefore, one can derive the existence and uniqueness conditions for problem directly from those for general VIs. Moreover, a number of additional existence and uniqueness results, essentially exploiting features of economic equilibrium models, have been obtained. On the other hand, there are a few approaches to compute economic equilibria. Most works are devoted to Scarf's simplicial labeling method, its modifications and extensions. Such methods are applicable for general economic equilibrium problems, however, need considerable additional storage, hence, they are implementable for rather small problems. Another approach is based on Newton's type methods. However, to ensure convergence, such methods, as well as the well-known tâtonnement process

$$p^{k+1} = \pi_{R^n_+}(p^k + \lambda_k q^k), q^k \in E(p^k), \lambda_k > 0. \tag{3.3}$$

need additional assumptions on E (or G). Indeed, the mapping E is not in general integrable, hence, such methods may fail even if $G = -E$ is not strictly monotone. In particular, process (3.3) is convergent if the following assumption hold:

$$\forall p \in R^n_+ \backslash P^*, \forall g \in G(p) : \langle g, p - p^* \rangle > 0, \tag{3.4}$$

where p^* is any equilibrium point. Assumption (3.4) can be treated as a variant of the revealed preference condition. Let us consider the problem of finding a point $p^* \in R^n_+$ such that

$$\forall p^* \in R^n_+, \forall g \in G(p) : \langle g, p - p^* \rangle \geq 0, \tag{3.5}$$

i.e., the dual problem to (3.2). It is clear that (3.5) is essentially weaker than (3.4), however, all the CR methods are convergent if there is a solution to problem (3.5). On the other hand, these methods are much more simpler than the simplicial type methods. Therefore, we intend to consider possible applications of CR methods to several economic equilibrium models. These models have been well documented in literature. In fact, if (3.5) (or (3.4)) holds, it is suitable to compute economic equilibrium in these models with the help of CR methods.

3.2 Transportation Equilibrium Problems

Flow equilibrium problems in transportation and communication systems constitute rather a new but broad and rapidly developing area of applications of variational inequalities. An essential feature of such problems consists mainly in the fact that they are determined on an oriented graph, each its arc being associated with some flow (for instance, traffic) and some expense (for instance, the time of motion, the time of delay, or cost ets), which depends on the values of arc flows. It is expected that increasing the value of flow for some arc increase the expense for this arc and perhaps for several neighbour arcs, which in turn implies redistribution of flows resulting in some equilibrium state.

There are a number of various formulations of transportation equilibrium problems. In this paper, we consider one of the most popular models, in which all the values of flow between origin-destination pairs are fixed. This model can be reduced to a variational inequality having a specific structure of constraints.

Let us given a graph with a finite set of nodes V and a set of oriented arcs D which join the nodes so that an arc $d = (i \to j)$ has the origin i and the destination j. Next, among all the pairs of nodes of the graph we extract a subset of pairs W of the form $w = (i \to j)$, where i is the origin node and j is the destination node. Besides, each pair $w \in W$ is associated with a positive number b_w which gives the flow demand from i to j. Denote by P_w the set of paths in the graph which connect the origin and destination, for the pair $w \in W$. Also, denote by x_p the path flow for the path p. Then, the feasible set of flows X can be defined as follows:

$$X = \{x \mid \sum_{p \in P_w} x_p = b_w, x_p \geq 0 \; p \in P_w; w \in W\}, \tag{3.6}$$

i.e.

$$X = \prod_{w \in W} X_w,$$

where

$$X_w = \{x \mid \sum_{p \in P_w} x_p = b_w, x_p \geq 0 \; p \in P_w\}.$$

If we set m to be the number of origin-destination pairs and l_w to be the number of paths joining the nodes of the pair w, then the total number of variables in this problem equals

$$n = l_1 \times \ldots \times l_m.$$

If the flow vector x is known, we can determine the value of the arc flow u_d for each arc $d \in D$:

$$u_d = \sum_{w \in W} \sum_{p \in P_w} \alpha_{pd} x_p,$$

where

$$\alpha_{pd} = \begin{cases} 1 & \text{if the arc } d \text{ belongs to the path } p, \\ 0 & \text{otherwise.} \end{cases}$$

If the values of arc flows are known, one can determine the value of expenses (costs) for each arc as follows:

$$t_d = T_d(u), \tag{3.7}$$

which in general depends on flows for other arcs and uses some mapping T that is defined in the space of flows. Then one can compute the value of expenses for each path p as follows:

$$g_p = G_p(x) = \sum_{d \in D} \alpha_{pd} t_d. \tag{3.8}$$

The feasible flow vector $x^* \in X$ is said to be an equilibrium vector if it satisfies the following conditions:

$$\forall q \in P_w, x_q^* > 0 \Longrightarrow G_q(x^*) = \min_{p \in P_w} G_p(x^*) \quad \text{for all } w \in W. \tag{3.9}$$

In other words, positive values of flow for any origin-destination pairs must correspond to paths with minimal costs. It is well known that the conditions (3.6) and (3.9) can be equivalently rewritten in the form of the variational inequality: Find a point $x^* \in X$ such that

$$\langle G(x^*), x - x^* \rangle \geq 0 \quad \forall x \in X, \tag{3.10}$$

where the inner product is defined in the n-dimensional space of paths joining all the nodes from W. Note that the fact of increasing (nondecreasing) delays along with increasing flows usually provides the monotonicity of the given mapping G, although there are such problems with nonmonotone cost mappings. It should be noted that the mapping G, defined by (3.7) and (3.8), is a potential one, i.e. presents the gradient of a scalar function, only under additional assumptions. Namely, in (3.7), delays for the arc d have not to be dependent on the flows for other arcs, or, equivalently, $t_d = T_d(u_d)$. In general, problem (3.6), (3.10) is not equivalent to an optimization problem, hence its solution cannot be found with analogs of the corresponding optimization methods. Therefore, to solve the variational inequality (3.6), (3.10) we can make use of iterative methods which are convergent under rather general assumptions, such as the CR methods.

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